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Noncommutative Field Theory

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We propose noncommutative space-time and a method to construct noncommutative field theory in terms of a covariant $\star$-product Moyal algebra and to study those physical and mathematical consequences. We consider noncommutative quantum electrodynamics. The prescription involves calculating the trace-like averaging procedure of noncommutative spacetime, leading to the nonlocal theory. From experimental data on testing the local theory it follows that $\theta \lesssim 7 \cdot 10^{-32} m^2$, where θ is the dimensionful scale of the tensor $\theta_{\mu\nu}$ characterizing noncommutative properties of spacetime arising from low-energy limit of string theories.

KEY WORDS: noncommutativity; vacuum polarization; space-time.

1. INTRODUCTION

We believe that a consistent relativistic quantum field theory of one-dimensional objects, i.e., the string theory (Polchinski, 1998) is a more complete theory with respect to the local quantum field theory (QFT) (Weinberg, 1995). One consequence of string theories is that the space-time coordinates satisfy nontrivial commutation relations (Ardalan *et al.*, 1998, 1999; Banks *et al.*, 1997; Connes *et al.*, 1998; de Wit *et al.*, 1988; Ishibashi *et al.*, 1997; Schomerus, 1999; Seiberg and Witten, 1999; Witten, 1996):

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \tag{1.1}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric tensor related to a background field $B_{\mu\nu}$ in the presence of a D-brane in string theories.

The noncommutative geometry (Connes, 1994; Gracia-Bondi *et al.*, 2000; Madore, 1999), the Euclidean field theories on some noncommutative geometric objects like sphere, plane, and cylinder (Grosse *et al.*, 1996a,b, 1997), the noncommutative analog of a Minkowski plane (Doplicher *et al.*, 1995; Snyder, 1947),

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and also Aharonov-Bohm and Casimir effects within the framework of noncommutative spaces (Chaichian *et al.*, 2001a,b) have been investigated.

Recently, noncommutative field theories (Chaichian *et al.*, 2001a,b; Huang, 2001; Kimura, 2001; Susskind, 2001) have been extensively studied, and the corresponding quantum mechanical problems have also received many attention (Athanasiu *et al.*, 1996; Bigatti and Susskind, 2000; Chaichian *et al.*, 2001a,b; Dunne *et al.*, 1990; Dunne and Jackiw, 1993; Duval and Horvathy, 2000; Floratos and Nicolai 2000; Gamboa *et al.*, 2000; Lukierski *et al.*, 1997; Morariu and Polychronakos, 2001; Nair, 2000; Nair and Polychronakos, 2000). There are the reviews in this field (Douglas and Nekrasov, 2001; Szabo, in press) and more recent works (Banerjee, 2002, and references cited therein Bolonek and Kosinski, 2002; Jonke and Meljanac, 2002). The explicit presence of the constant $\theta_{\mu\nu}$ in (1.1) violates Lorentz invariance. It was shown that in such noncommutative Minkowski spaces, the ultraviolate divergences of the QFT persist (Chaichian *et al.*, 2000a,b,c; Filk, 1996) and unitarity and causality (Chaichian *et al.*, 2000a,b,c; Gomis and Mehen, 2000; Seiberg *et al.*, 2000) are also broken.

Thus, we see that an attempt to construct self-consistent QFT directly on noncommutative Minkowski space (1.1) encounters difficulties due to violation of basic physical principles like Lorentz and gauge inveriances, unitarity, and causality. Here we try to study this problem. Main assumption is that in string theories spacetime has more than four dimensions with the additional noncommutative ones so highly curved as to be undetectable at current energies. While noncommutative four-dimensional fields become as residual or averaging effects at low-energy limit of string theories with noncommutative spacetime satisfying commutation relations (1.1).

The noncommutative models defined by (1.1) can be realized in terms of a $\star$-product. The commutative algebra A_0 of functions with the usual product $(fg)(x) = f(x)g(x)$ is replaced by the $\star$-product Moyal algebra:

$$(f \star g)(x) = \exp\left[\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right]|_{x=y} = f(x)g(x) + \frac{i}{2}\{f, g\}(x) + O(\theta^2) \quad (1.2)$$

where $\{f, g\} = \theta_{\mu\nu}(\partial_\mu f)(\partial_\nu g)$ is the Poisson bracket associated with $\theta_{\mu\nu}$. Such associative $\star$-products have been proved to exist as a formal power series for any Poisson bracket $\{f, g\} = \theta_{\mu\nu}(x)(\partial_\mu f)(\partial_\nu g)$, with a most general x-dependent $\theta_{\mu\nu}(x)$ (Caetano and Felder, 1999; Kontsevich, 1997). To solve the problems of the summability and unitarizability, we would like to act as follows:

1. Let $\theta_{\mu\nu}$ in (1.1) be constant defined by the formula

$$\theta_{\mu\nu} = \frac{1}{i}\theta(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{1}{i}\sigma_{\mu\nu}\theta \quad (1.3)$$

in any d-dimensional spacetime with the Dirac γ_μ-matrices.

2. The Moyal $\star$-product is replaced by the covariant $(\star)_c$-product:

$$(f(\star)_c g)(x) = \frac{1}{N(d)} \mathrm{Tr} \left\{ \exp\left[\frac{1}{2} \theta \sigma_{\mu\nu} \partial_{x_\mu} \partial_{y_\nu} \right] \right\} f(x) g(y)|_{x=y} \tag{1.4}$$

Here all variables and trace are taken in d-dimensional spacetime with

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \qquad g_{\mu\nu} = d \tag{1.5}$$
$$\mathrm{Tr}\,(I) = N(d)$$

where N is a regular function of d only and $N(4) = 4$.

Physical meaning of the covariant $(\star)_c$-product (1.4) is that noncommutative properties of spacetime take place in the d-dimensional case and our usual four-dimensional spacetime and physical fields on it are defined as residual or averaging procedure obtained by taking trace of γ_μ-matrices.

Our next goal is to show that this prescription allows us to construct noncommutative quantum field theory free from the above-mentioned difficulties in the context of noncommutative spacetime (1.1). Outline of this work is as follows. In Section 2, we modify definition of the Moyal $\star$-product in any d-dimensional spacetime and calculate trace of its noncommutativity. Section 3 deals with free fields and their commutation relations, the Pauli–Jordan and Green functions in the noncommutative spacetime (1.1). The next four Sections 4–7 are devoted to the construction of the noncommutative quantum electrodynamics and to the calculation of the vacuum polarization, the anomalous magnetic moment of leptons and the electron self-energy by means of noncommutative algebra of field operator functions on $\mathcal{R}^4$. Finally, we estimate restriction on the dimensionaful scale θ of the tensor in (1.1). In Section 8 we study casuality and unitarity conditions for the $S_\star$-matrix of the noncommutative field theory. Some geometrical and physical consequences of the noncommutative theory are considered in Section 9.

2. REDEFINITION OF THE MOYAL $\star$-PRODUCT AND TRACE OF NONCOMMUTATIVITY OF SPACE-TIME

In this section, we shall first describe the Moyal $\star$-product on any d-dimensional spacetime $\mathcal{R}^d$. The commutative algebra A_0 of functions on $\mathcal{R}^d$ is formed by functions of the form:

$$f(x) = \frac{1}{(2\pi)^d} \int d^d k\, e^{ikx}\, \tilde{f}(k) \tag{2.1}$$

where

$$k \cdot x = g_{\mu\nu} x^\mu x^\nu = -x^0 k^0 + k^1 x^1 + \cdots + k^{d-1} \cdot x^{d-1}.$$

Then, the Moyal product is defined as

$$(f \star g)(x) = \frac{1}{(2\pi)^{2d}} \int d^d k_1 \, d^d k_2 \; \tilde{f}(k_1)\tilde{g}(k_2)\, e^{i(k_1+k_2)x} \cdot e^{-\frac{i}{2}\theta_{\mu\nu} k_1^\mu k_2^\mu} \tag{2.2}$$

where $\theta_{\mu\nu} = \frac{1}{i}\theta\sigma_{\mu\nu}$, θ-constant, $\sigma_{\mu\nu}$-antisymmetric. This formula defines the corresponding noncommutative algebra of functions A_θ on $\mathcal{R}^d$.

Alternatively, one can start from an operator algebra generated by the Hermitian operators $\hat{x}_\mu$ and $\hat{x}_\nu$, satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$$

The corresponding noncommutative algebra A_θ can be given as the algebra of operators of the form

$$f(\hat{x}) = \frac{1}{(2\pi)^d} \int d^d k \; \tilde{f}(k)\, e^{ik\hat{x}} \tag{2.3}$$

where

$$ik\hat{x} = g_{\mu\nu}\hat{x}^\mu k^\nu$$

It can be seen easily that the product in the operator algebra (2.3) possesses an expansion in powers of θ, exactly corresponding to the Moyal product.

One can see that the assumption (1.3) leads to the change

$$(f \star g)(x) = \frac{1}{(2\pi)^{2d}} \int d^d k_1 \, d^d k_2 \left\{ e^{-\frac{\theta}{2}\cdot\sigma_{\mu\nu} k_1^\mu k_2^\nu} \right\} \tilde{f}(k_1)\tilde{g}(k_2)\, e^{i(k_1+k_2)x} \tag{2.4}$$

where

$$\sigma_{\mu\nu} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$$

Now the covariant $(\star)_c$-product (1.4) takes the form

$$\begin{aligned}(f(\star)_c g)(x) = \frac{1}{(2\pi)^{2d}} \int d^4 k_1 \, d^4 k_2 \frac{1}{N(d)} \mathrm{Tr}\left\{ e^{-\frac{\theta}{2}\cdot\sigma_{\mu\nu} k_1^\mu k_2^\mu} \right\} \\ \times \tilde{f}(k_1)\tilde{g}(k_2)\, e^{i(k_1+k_2)x} \end{aligned} \tag{2.5}$$

Our next goal is to calculate trace in (2.5) for any order in θ. For this purpose, we use algebra of γ_μ-matrices in d-dimensional spacetime, defined by t' Hooft and Veltman (1972).

Expanding exponential in (2.5) by the Taylor series and calculating trace for each terms, one gets

1. $\frac{1}{2}\theta \, \mathrm{Tr}\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) k_1^\mu k_2^\nu = N(d)\theta[(k_1 \cdot k_2) - (k_2 \cdot k_1)] = 0$

2.
$$\frac{1}{2!}\frac{\theta^2}{2^2}\mathrm{Tr}\,[\sigma_{\mu\nu}\sigma_{\rho\sigma}]k_1^{\mu}k_2^{\nu}\cdot k_1^{\rho}k_2^{\sigma}$$
$$= \frac{1}{2!}\theta^2 N(d)[g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}]\left(k_1^{\mu}k_2^{\nu}\cdot k_1^{\rho}k_2^{\sigma}\right)$$
$$= \frac{1}{2}\left[(k_1\cdot k_2)^2 - k_1^2k_2^2\right]\cdot\theta^2 N(d) \qquad (2.6)$$

3.
$$\frac{1}{3!}\frac{\theta^3}{2^3}\mathrm{Tr}\,[\sigma_{\mu\nu}\sigma_{\rho\sigma}\sigma_{\lambda\chi}]k_1^{\mu}k_2^{\nu}k_1^{\rho}k_2^{\sigma}k_1^{\lambda}k_2^{\chi}$$
$$= \frac{1}{3!}\theta^3\cdot N(d)\{-g_{\mu\rho}[g_{\nu\chi}g_{\sigma\lambda} - g_{\nu\lambda}g_{\sigma\chi}]$$
$$+ g_{\mu\sigma}[g_{\nu\chi}g_{\rho\lambda} - g_{\nu\lambda}g_{\rho\chi}] - g_{\mu\lambda}[g_{\nu\rho}g_{\sigma\chi} - g_{\nu\sigma}g_{\rho\chi}]$$
$$+ g_{\mu\chi}[g_{\nu\rho}g_{\sigma\lambda} - g_{\nu\sigma}g_{\rho\lambda}]k_1^{\mu}k_2^{\nu}k_1^{\rho}k_2^{\sigma}k_1^{\lambda}k_1^{\chi}$$
$$= \frac{1}{3!}\theta^3\cdot N(d)\left\{-k_1^2\left(k_2^2(k_1\cdot k_2) - (k_2\cdot k_1)k_2^2\right)\right.$$
$$+ (k_1\cdot k_2)\left(k_1^2k_2^2 - (k_1\cdot k_2)^2\right) - k_1^2\left(k_2^2(k_1\cdot k_2)\right.$$
$$\left.\left. - (k_2\cdot k_1)k_2^2\right) - (k_1\cdot k_2)\left(k_1^2k_2^2 - (k_1\cdot k_2)^2\right)\right\} \equiv 0 \qquad (2.7)$$

To see more or less approximate form of the Taylor series, we calculate yet one trace of $\sigma_{\mu\nu}\sigma_{\rho\sigma}\sigma_{\lambda\chi}\sigma_{\alpha\beta}$-product terms. After a straightforward but tedious calculation, this trace then becomes

4.
$$\frac{1}{4!}\frac{\theta^4}{2^4}\mathrm{Tr}\,[\sigma_{\mu\nu}\sigma_{\rho\sigma}\sigma_{\lambda\chi}\sigma_{\alpha\beta}]k_1^{\mu}k_2^{\nu}k_1^{\rho}k_2^{\sigma}k_1^{\lambda}k_2^{\chi}k_1^{\alpha}k_2^{\beta}$$
$$= \frac{1}{4!}\theta^4 N(d)\{-g_{\mu\rho}[g_{\nu\sigma}(g_{\lambda\beta}g_{\lambda\alpha} - g_{\lambda\alpha}g_{\chi\beta}) - g\nu\lambda(g_{\sigma\beta}g_{\chi\alpha} - g_{\sigma\alpha}g_{\chi\beta})$$
$$+ g_{\nu\chi}(g_{\sigma\beta}g_{\lambda\alpha} - g_{\sigma\alpha}g_{\lambda\beta}) - g_{\nu\alpha}(g_{\sigma\lambda}g_{\chi\beta} - g_{\sigma\chi}g_{\lambda\beta})$$
$$+ g_{\nu\beta}(g_{\sigma\lambda}g_{\chi\alpha} - g_{\sigma\chi}g_{\lambda\alpha})] + g_{\mu\sigma}[g_{\nu\rho}(g_{\lambda\beta}g_{\chi\alpha} - g_{\lambda\alpha}g_{\chi\beta})$$
$$- g_{\nu\lambda}(g_{\rho\beta}g_{\chi\alpha} - g_{\rho\alpha}g_{\chi\beta}) + g_{\nu\chi}(g_{\rho\beta}g_{\lambda\alpha} - g_{\rho\alpha}g_{\lambda\beta})$$
$$- g_{\nu\alpha}(g_{\rho\lambda}g_{\chi\beta} - g_{\rho\chi}g_{\lambda\beta}) + g_{\nu\beta}(g_{\rho\lambda}g_{\chi\alpha} - g_{\rho\chi}g_{\lambda\alpha})]$$
$$- g_{\mu\lambda}[g_{\nu\rho}(g_{\sigma\beta}g_{\chi\alpha} - g_{\sigma\alpha}g_{\chi\beta}) - g_{\nu\sigma}(g_{\rho\beta}g_{\chi\alpha} - g_{\rho\alpha}g_{\chi\beta})$$
$$+ g_{\nu\chi}(g_{\rho\beta}g_{\sigma\alpha} - g_{\rho\alpha}g_{\sigma\beta}) - g_{\nu\alpha}(g_{\rho\beta}g_{\sigma\chi} - g_{\rho\chi}g_{\sigma\beta})$$
$$+ g_{\nu\beta}(g_{\rho\alpha}g_{\sigma\chi} - g_{\rho\chi}g_{\sigma\alpha})] + g_{\mu\chi}[g_{\nu\rho}(g_{\sigma\beta}g_{\lambda\alpha} - g_{\sigma\alpha}g_{\lambda\alpha})$$
$$- g_{\nu\sigma}(g_{\rho\beta}g_{\lambda\alpha} - g_{\rho\alpha}g_{\lambda\beta}) + g_{\nu\lambda}(g_{\rho\beta}g_{\sigma\alpha} - g_{\rho\alpha}g_{\sigma\beta})$$
$$- g_{\nu\alpha}(g_{\rho\beta}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\beta}) + g_{\nu\beta}(g_{\rho\alpha}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\alpha})]$$
$$- g_{\mu\alpha}[g_{\nu\rho}(g_{\sigma\lambda}g_{\chi\beta} - g_{\sigma\chi}g_{\lambda\beta}) - g_{\nu\sigma}(g_{\rho\lambda}g_{\chi\beta} - g_{\rho\chi}g_{\lambda\beta})$$

$$+ g_{\nu\lambda}(g_{\rho\beta}g_{\sigma\chi} - g_{\rho\chi}g_{\sigma\beta}) - g_{\nu\chi}(g_{\rho\beta}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\beta})$$
$$+ g_{\nu\beta}(g_{\rho\chi}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\chi})] + g_{\mu\beta}[g_{\nu\rho}(g_{\sigma\lambda}g_{\chi\alpha} - g_{\sigma\chi}g_{\lambda\alpha})$$
$$- g_{\nu\sigma}(g_{\rho\lambda}g_{\chi\alpha} - g_{\rho\chi}g_{\lambda\alpha}) + g_{\nu\lambda}(g_{\rho\alpha}g_{\sigma\chi} - g_{\rho\chi}g_{\sigma\alpha})$$
$$- g_{\nu\chi}(g_{\rho\alpha}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\alpha}) + g_{\nu\alpha}(g_{\rho\chi}g_{\sigma\lambda} - g_{\rho\lambda}g_{\sigma\chi})]\}$$
$$\times k_1^\mu k_2^\nu k_1^\rho k_2^\sigma k_1^\lambda k_2^\chi k_1^\alpha k_2^\beta = \frac{\theta^4}{4!} N(d)\left[(k_1 \cdot k_2)^2 - k_1^2 k_2^2\right]^2 \tag{2.8}$$

Thus, one can see that trace of expression

$$I_\theta = \frac{1}{N(d)} \mathrm{Tr}\left[e^{-\frac{1}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu}\right] \tag{2.9}$$

due to noncommutativity of spacetime is approximated by the elementary function

$$I_\theta = \cosh\left(\theta\sqrt{(k_1 \cdot k_2)^2 - k_1^2 k_2^2}\right) \tag{2.10}$$

at least up to desired order of θ^5.

In accordance with the formulas (2.9) and (2.10) the covariant $(\star)_c$-product (2.5) acquires the form in the momentum space

$$(f(\star)_c g)(x) = \frac{1}{(2\pi)^{2d}} \int d^d k_1\, d^d k_2 \cosh\left(\theta\sqrt{(k_1 \cdot k_2)^2 - k_1^2 k_2^2}\right)$$
$$\times\ \tilde{f}(k_1)\tilde{g}(k_2)\, e^{i(k_1+k_2)x} \tag{2.11}$$

or in the coordinate representation

$$(f(\star)_c g)(x) = \cosh\left(\theta\sqrt{\left(\partial_x^\mu \cdot \partial_\mu^y\right)^2 - \Box_x \cdot \Box_y}\right) f(x)g(y)|_{y=x} \tag{2.12}$$

where

$$\partial_\mu^x = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right) \quad \text{and} \quad \Box_x = \frac{\partial^2}{\partial x^{02}} - \vec{\nabla}^2$$

We see that residual effect resulting from the averaging procedure (taking trace) on the noncommutative spacetime leads to the nonlocal commutative algebra A_θ of functions with the mixing product like (2.11) and (2.12). We call this type of correlation nonlocal product.

3. OPERATOR PRODUCT OF NONCOMMUTATIVE QUANTIZED FIELDS

In this section, we investigate commutation relations and Green functions of noncommutative quantized fields. We consider here only scalar particles. The

$\star$-product of quantized scalar fields $\varphi(x)$ defines as

$$\varphi(x) \star \varphi(x) = \frac{1}{(2\pi)^8} \int d^4k_1\, d^4k_2\, e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu} \tilde{\varphi}(k_1)\tilde{\varphi}(k_2)\, e^{i(k_1+k_2)x} \tag{3.1}$$

and its covariant $(\star)_c$-product reads

$$(\varphi(x)(\star)_c\varphi(y)) = \cosh\left(\theta\sqrt{\left(\partial_x^\mu \cdot \partial_\mu^y\right)^2 - \Box_x \cdot \Box_y}\right)\varphi(x)\varphi(y) \tag{3.2}$$

The commutator of noncommutative field operators $\varphi(x)$ takes the form

$$\begin{aligned} D_\theta(x-y) &= \varphi(x)(\star)_c\varphi(y) - \varphi(y)(\star)_c\varphi(x) \\ &= \cosh\left(\theta\sqrt{\left(\partial_x^\mu \cdot \partial_\mu^y\right)^2 - \Box_x \cdot \Box_y}\right)\Delta(x-y) \end{aligned} \tag{3.3}$$

where

$$\Delta(x) = \frac{i}{(2\pi)^3} \int d^4k\, \varepsilon(k^0)\delta(k^2+m^2)\, e^{-ikx}$$

is the Pauli-Jordan function of the scalar particle. From this it follows diretly,

$$D_\theta(x-y) = \Delta(x-y) \tag{3.4}$$

since $\cos 0 = 1$.

Similar definition for the Green function of the scalar particle holds

$$\begin{aligned} D_\theta^c(x-y) &= \langle 0|T[\varphi(x)(\star)_c\varphi(y)]|0\rangle = \overbrace{\varphi(x)(\star)_c\varphi(y)} \\ &= \cosh\left(\theta\sqrt{\left(\partial_x^\mu \cdot \partial_\mu^y\right)^2 - \Box_x \cdot \Box_y}\right)\Delta^c(x-y) \equiv \Delta^c(x-y) \end{aligned} \tag{3.5}$$

where $\Delta^c(x-y)$ is the usual local Green function of the scalr field. It is obvious that all 2-point functions of noncommutative quantized fields $\varphi(x)$ coincide with their local ones:

$$D_\theta^\pm(x) = \Delta^\pm(x), \quad D_\theta^{\text{adv,ret}}(x) = \Delta^{\text{adv,ret}}(x), \quad D_\theta^c(x) = \Delta^c(x) \tag{3.6}$$

We will show below that nontrivial contributions due to noncommutativeity of spacetime appear only in the case of interacting fields. In other words, interaction Lagrangians and Feynman diagrams are changed in accordance with the definition of the $\star$-product of noncommutative quantized fields. Let us consider, for example, the $\varphi^3(x)$-theory. Its Lagrangian takes the form

$$\mathcal{L}_\theta = \frac{1}{3!} g : \varphi(x) \star \varphi(x) \star \varphi(x) : \tag{3.7}$$

S_θ-matrix for this noncommutative theory defines as

$$S_\theta = T \star \exp\left\{ i \int d^4x\, \mathcal{L}_\theta(x) \right\} \tag{3.8}$$

We write the S_θ-matrix in the momentum representation. According to the Wick theorem after taking the normal form, term of n-th order of the scattering matrix can be expressed by sum of terms

$$\int dx_1 \ldots \int dx_n \, K_\theta^\star(x_1, \ldots, x_n) : \star \cdots \star \varphi(x_i) \star \cdots \star \varphi^*(x_j) \star \cdots \star : \tag{3.9}$$

where coefficient functions $K_\theta^\star(x, \ldots, x_n)$ correspond to internal lines of the Feynman diagrams and are formed by $\star$-products of the Green functions:

$$\star\Delta(x_i - x_j) \star \Delta(x_j - x_k) \star \cdots \tag{3.10}$$

While a normal $\star$-product

$$: \star \cdots \star \varphi(x_j) \star \cdots \star \varphi^\star(x_i) \star \cdots \star : \tag{3.11}$$

contains free operator fields corresponding to external lines of diagrams.

The structure of the matrix elements of the S_θ-matrix has the general form

$$\Phi^*_{\ldots p' \ldots} \star S_\theta \star \Phi_{\ldots p \ldots} \tag{3.12}$$

where initial $\Phi_{\ldots p \ldots}$ and final $\Phi^*_{\ldots p' \ldots}$-states are formed by means of creation operators $a^+(\mathbf{k})$:

$$\Phi_{\ldots k \ldots} = a_1^+(\mathbf{k}_1) a_2^+(\mathbf{k}_2) \ldots a_n^+(\mathbf{k}_n) \Phi^0 \tag{3.13}$$

The matrix element

$$\Phi^*_{\ldots p' \ldots} : \star \cdots \star \varphi(x_j) \star \cdots \star \varphi^*(x_i) \star \cdots \star : \Phi_{\ldots p \ldots} \tag{3.14}$$

is represented in the form of the $\star$-products resulting from commutations between operators

$$\varphi^-(x_j) \quad \text{with} \quad a^+(\mathbf{k}) = \int dk^0 \, \theta(k^0) a^{*+}(k) \delta(k^2 + m^2) \sqrt{2k^0}$$

and

$$\varphi^+(x_i) \quad \text{with} \quad a^{*-}(\mathbf{k}) = \int dk^0 \, \theta(k^0) a^{*-}(k) \delta(k^2 + m^2) \sqrt{2k^0}$$

Thus, after carrying out commutations, the matrix element (3.14) is indeed expressed by the $\star$-product:

$$(2\pi)^{-3/2} e^{-ik_1 x_1} \star (2\pi)^{-3/2} e^{-ik_2 x_2} \star \cdots \star (2\pi)^{-3/2} e^{-ik_1' x_1'} \star (2\pi)^{-3/2} e^{-ik_2' x_2'} \star \cdots$$

of plane waves.

It is easy to realize the $\star$-product S_θ*-matrix* elements in Feynman diagrams. For example, the following matrix element

$$\sim \int d^4x \int d^4y \, e^{-ipx} \star \Delta(x - y) \star \Delta(y - x) \star e^{ip'y} \tag{3.15}$$

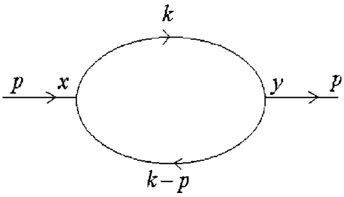


Fig. 1. Primitive Feynman diagram in noncommutative scalar $\frac{1}{3!}g.\varphi(x)\star\varphi(x)\star\varphi(x)$:-theory.

corresponds to the Feynman diagram shown in Fig. 1, in x-space. In the momentum space it takes the natural form

$$\Pi(p) = \frac{g^2}{(2\pi)^4}\frac{1}{i}\int d^4k \times \exp\left[\frac{1}{2}\theta\sigma_{\mu\nu}p^\mu k^\nu + \frac{1}{2}\theta\sigma_{\rho\sigma}k^\rho(k-p)^\sigma - \frac{1}{2}\theta\sigma_{\alpha\beta}p^\alpha(k-p)^\beta\right] \times \frac{1}{k^2+m^2-i\varepsilon}\cdot\frac{1}{(k-p)^2+m^2-i\varepsilon} \tag{3.16}$$

Taking into account the identity $\sigma_{\mu\nu}p^\mu p^\nu = \sigma_{\mu\nu}k^\mu k^\nu = 0$ and calculating trace of the γ_μ-matrices, one gets

$$\Pi_\theta(p) = \text{Tr}\,\Pi(p) = \frac{g^2}{(2\pi)^4}\frac{1}{i}\int d^4k \cosh\left(\theta\sqrt{(k\cdot p)^2 - p^2k^2}\right) \times \frac{1}{k^2+m^2-i\varepsilon}\frac{1}{(k-p)^2+m^2-i\varepsilon} \tag{3.17}$$

Here rapidly oscillation and sign variable function $\cosh i\theta x$ can be expressed in terms of the Mellin representation:

$$\cosh\left(\theta\sqrt{(k\cdot p)^2 - p^2k^2}\right) = \frac{1}{2i}\int_{-\beta+i\infty}^{-\beta-i\infty} d\xi\,\frac{1}{\sin\pi\xi}\,\frac{\theta^{2\xi}[-(k\cdot p)^2 + p^2k^2]^\xi}{\Gamma(1+2\xi)} \tag{3.18}$$

where $\beta > 0$ is any number. Then, we see that after passing to the Euclidian or a Wick rotation, whole integral (3.17) is converged

$$\int_0^\infty \frac{dk\cdot k^3}{k^4}[k^2]^{-\beta} < 0, \qquad k = \sqrt{k^2}$$

In next sections we will study Feynman diagrams in the noncommutative quantum electrodynamics.

4. NONCOMMUTATIVE QUANTUM ELECTRODYNAMICS, THE LAGRANGIAN DENSITY, COUNTERTERMS, AND GAUGE INVARIANCE

In this section, we shall define the Lagrangian density and proceed to carry out some techniques with the $\star$-product to show gauge invariance of the theory of charged leptons that interact with the electromagnetic field in non-commutative spacetime. For noncommutative spinor (electron) and photon fields the Lagrangian density is taken in the form of the $\star$-product:

$$\mathcal{L} = -\frac{1}{4}F_B^{*\mu\nu} \star F_{B\mu\nu} - \bar{\psi}_B(x) \star \left[\gamma_\mu \left(\partial^\mu + i\, e_B A_B^\mu(x)\star\right) + m_B\right] \psi_B(x) \quad (4.1)$$

where

$$F_B^{\mu\nu} \equiv \partial^\mu A_B^\nu - \partial^\nu A_B^\mu - i\, e_B(A^\mu \star A^\nu - A^\nu \star A^\mu)$$

and ψ_B are the bare (unrenormalized) noncommutative fields of the photon and electron, and $-e_B$ and m_B are the bare charge and mass of the electron. As in the local field theory, we introduce renormalized fields, charge, and mass:

$$\psi \equiv Z_2^{-1/2}\psi_B, \qquad A^\mu \equiv Z_3^{-1/2} A_B^\mu \quad (4.2)$$

$$e \equiv Z_2^{1/2} e_B, \qquad m \equiv m_B + \delta m \quad (4.3)$$

with the constants Z_2, Z_3, and δm. As usually, the Lagrangian may then be written in terms of renormalized noncommutative quantities

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \quad (4.4)$$

where

$$\mathcal{L}_0 = -\frac{1}{4}F^{*\mu\nu} \star F_{\mu\nu} - \bar{\psi} \star [\gamma_\mu \partial^\mu + m]\psi \quad (4.5)$$

$$\mathcal{L}_1 = -i\, e A_\mu(x) \star \bar{\psi}(x) \star \gamma^\mu \psi(x) \quad (4.6)$$

and $\mathcal{L}_2$ is defined as a sum of "counterterms"

$$\mathcal{L}_2 = -\frac{1}{2}(Z_3 - 1)F^{*\mu\nu} \star F_{\mu\nu} - (Z_2 - 1)\bar{\psi} \star [\gamma_\mu \partial^\mu + m]\psi$$
$$+ Z_2 \delta m \star \bar{\psi} \star \psi - i\, e(Z_2 - 1)A_\mu(x) \star \bar{\psi}(x) \star \gamma^\mu \psi(x) \quad (4.7)$$

Notice that all of the terms in $\mathcal{L}_2$ are of second order and higher order in e, and that these terms ensure to cancel the ultraviolet divergences that arise from loop graphs in the noncommutative quantum electrodynamics.

We shall now show that the Lagrangian density (4.1) is invariant under the noncommutative analog of gauge transformations with the $\star$-product:

$$\psi_B(x) \to e^{i\lambda(x)} \star \psi_B'(x), \quad \bar{\psi}_B'(x) \to \bar{\psi}_B(x) \star e^{-i\lambda(x)} \quad (4.8)$$

and

$$A_B^\mu(x) \to \left[A_B'^\mu(x) - \frac{1}{e_B} \frac{\partial\lambda(x)}{\partial x^\mu} \right]_*$$
$$\to e^{i\lambda(x)} \star A_B'^\mu(x) \star e^{-i\lambda(x)} - \frac{1}{e_B} e^{i\lambda(x)} \star \frac{\partial\lambda(x)}{\partial x^\mu} \star e^{-i\lambda(x)} \tag{4.9}$$

with an arbitrary function $\lambda(x)$.

Taking into account formulas (4.8) and making use of the differentiation

$$\partial^\mu \psi_B = e^{i\lambda(x)} \star i \frac{\partial\lambda(x)}{\partial x^\mu} \star \psi' + e^{i\lambda(x)} \star \frac{\partial}{\partial x^\mu} \psi'$$

one gets

$$\begin{aligned} \bar{\psi}_B(x) \star [\gamma_\mu \partial^\mu + m] \psi_B(x) \to -\bar{\psi}'_B(x) \star e^{-i\lambda(x)} \star \\ \times \left[\gamma_\mu \left(e^{i\lambda(x)} \star i \frac{\partial\lambda(x)}{\partial x^\mu} \star \psi'_B(x) + e^{i\lambda(x)} \star \frac{\partial}{\partial x^\mu} \psi'(x) \right.\right. \\ \left.\left. + i\, e_B A_B^\mu(x) \star e^{i\lambda(x)} \star \psi'_B(x) \right) + m_B\, e^{i\lambda(x)} \star \psi'_B(x) \right] \end{aligned} \tag{4.10}$$

where the noncommutative photon field $A_B^\mu(x)$ is transformed by the formula (4.9). Substitute it into (4.10) for $A_B^\mu(x)$-field and prove its gauge invariance

$$\begin{aligned} -\bar{\psi}_B(x) \star \left[\gamma_\mu \left(\partial^\mu + i\, e_B A_B^\mu(x) \star \right) + m \right] \psi_B(x) \to -\bar{\psi}'_B(x) \star e^{-i\lambda(x)} \star \\ \times \left[\gamma_\mu \left(e^{i\lambda(x)} \star i \frac{\partial\lambda(x)}{\partial x^\mu} \star \psi'_B(x) + e^{i\lambda(x)} \star \frac{\partial}{\partial x^\mu} \psi'(x) \right.\right. \\ + i\, e_B\, e^{i\lambda(x)} \star A'^\mu(x) \star e^{-i\lambda(x)} \star e^{i\lambda(x)} \star \psi'(x) \\ \left.\left. -\ i\, e^{i\lambda(x)} \star \frac{\partial\lambda(x)}{\partial x^\mu} \star e^{-i\lambda(x)} \star e^{i\lambda(x)} \star \psi'_B(x) \right) \right] + m_B\, e^{i\lambda(x)} \star \psi'(x) \\ = \bar{\psi}'(x) \star \left[\gamma_\mu \left(\partial^\mu + i\, e_B A_B'^\mu(x) \star \right) + m \right] \psi'_B(x) \end{aligned} \tag{4.11}$$

where we have used the identity $e^{-i\lambda(x)} \star e^{-i\lambda(x)} = 1$.

It is natural that the field strength

$$F_B^{\mu\nu} = \partial^\mu A_B^\nu - \partial^\nu A_B^\mu - i\, e_B \left(A_B^\nu \star A_B^\mu - A_B^\mu \star A_B^\nu \right) \tag{4.12}$$

in (4.1) and (4.5) is given by a non-Abelian formula which is invariant with respect to the transformation (4.9), where

$$\frac{\partial\lambda}{\partial x^\mu} \star \frac{\partial\lambda}{\partial x^\nu} = \frac{\partial\lambda}{\partial x^\nu} \star \frac{\partial\lambda}{\partial x^\mu}$$

and

$$\frac{\partial\lambda}{\partial x^\nu} \star A^\mu \neq \frac{\partial\lambda}{\partial x^\mu} \star A^\nu$$

If the commutator

$$[\partial_\nu\lambda, \star A^\mu] = \frac{\partial\lambda}{\partial x^\nu} \star A^\mu - A^\mu \star \frac{\partial\lambda}{\partial x^\mu}$$

is equal to zero, then the last term in (4.12) is disappeared for noncommutative photon fields $A^\theta_\mu(x)$.

One can verify that the field strength (4.12) is valid for a more general case when both the transformation function $\lambda(x)$ and the gauge fields $A^\mu(x)$ possess an internal symmetry defined by their matrix values $\hat\lambda(x)$ and $\hat A^\mu(x)$. However the nonlinear term in (4.12)

$$\left(A^\nu_B \star A^\mu_B - A^\mu_B \star A^\nu_B\right) = \frac{1}{(2\pi)^8}\int d^4p\, d^4q\, e^{-i(p+q)x} \times\left[e^{\frac{1}{2}\theta\sigma_{\rho\sigma}q^\rho p^\sigma} - e^{\frac{1}{2}\theta\sigma_{\alpha\beta}q^\alpha p^\beta}\right]\tilde A^\nu(q)\tilde A^\mu(p)$$

goes to zero when we will use the covariant $(\star)_c$-product instead of the usual Moyal $\star$-product between them.

Below we use the following type of $\star$-products and those covariant versions:

$$f(x)\star g(x) = \frac{1}{(2\pi)^{2d}}\int d^dp\, d^dq\, e^{-\frac{1}{2}\theta\sigma_{\mu\nu}p^\mu q^\nu}\, e^{i(p+q)x}\,\tilde f(p)\tilde g(q),$$

$$f(x)\star F(x-y)\star g(x) = \frac{1}{(2\pi)^{3d}}\int d^dp\, d^dk\, d^dq\, e^{-\frac{1}{2}\theta\sigma_{\mu\nu}p^\mu k^\nu + \frac{1}{2}\theta\sigma_{\rho\sigma}k^\rho q^\sigma} \times e^{ipx+iqy+ik(x-y)}\,\tilde f(p)\tilde F(k)\tilde g(q)$$

and

$$f(x)\star F(x-y)\star G(g-x)\star g(x) = \frac{1}{(2\pi)^{4d}}\int d^dp\, d^dk_1\, d^dk_2\, d^dq \times \exp\left[-\frac{1}{2}\theta\sigma_{\mu\nu}p^\mu k_1^\nu + \frac{1}{2}\theta\sigma_{\chi\lambda}k_1^\chi k_2^\lambda - \frac{1}{2}\theta\sigma_{\rho\sigma}k_2^\rho q^\sigma\right] \times e^{ipx+iqy+ik_1(x-y)+ik_1(y-x)}\,\tilde f(p)\tilde F(k_1)\tilde G(k_2)\tilde g(q)$$

and so on, where $\sigma_{\mu\nu} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$.

We now turn to calculate Feynman diagrams in the noncommutative quantum electrodynamics (NQED) defined by the Lagrangians (4.5), (4.6), and (4.7).

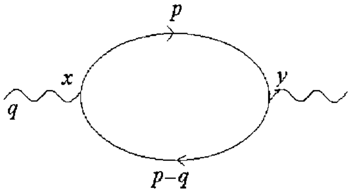


Fig. 2. The one-loop diagram for the vacuum polarization in noncommutative quantum electrodynamics.

5. VACUUM POLARIZATION

The Feynman rule in NQED is same as in the local QED with only difference that in vertices of diagrams factors $\exp(-\frac{1}{2}\theta\sigma_{\mu\nu}p^{\mu}q^{\nu})$ are arisen from the products between plane waves in external lines and the Green functions in internal lines.

In the coordinate space, the matrix element of the S_θ-matrix in NQED, corresponding to the diagram in Fig. 2 has the form

$$
\begin{aligned}
&-i\ :A_\mu(x)\star\{-i\,e^2\gamma^\mu S(x-y)\star\gamma^\nu S(y-x)\}\star A_\nu(y)\\
&:=-i:A_\mu(x)\star\Pi^{\mu\nu}(x-y)\star A_\nu(y):
\end{aligned}
$$

where

$$
\Pi^{\mu\nu}(x-y)=-i\,e^2\,\mathrm{Tr}\,\{\gamma^\mu\star S(x-y)\gamma^\mu\star S(y-x)\} \tag{5.1}
$$

Here the $\star$-product

$$
e^{-iqx}\star e^{-ip(x-y)}\star e^{-i(p-q)(y-x)}\star e^{iqy}
$$

leads to the form factor in the momentum space

$$
\begin{aligned}
&\exp\left[-\frac{1}{2}\theta\sigma_{\mu\nu}q^\mu p^\nu+\frac{1}{2}\theta\sigma_{\chi\lambda}q^\chi(p-q)^\lambda+\frac{1}{2}\theta\sigma_{\rho\delta}(p-q)^\rho p^\delta\right]\\
&=\exp\left(-\frac{1}{2}\theta\sigma_{\mu\nu}p^\nu q^\mu\right)
\end{aligned} \tag{5.2}
$$

and therefore the vacuum polarization (5.1) in p-space reads

$$
\Pi^{\rho\sigma}(q)=\frac{-i\,e^2}{(2\pi)^4}\int d^4p\,e^{-\frac{1}{2}\theta\sigma_{\mu\nu}p^\nu q^\mu}\times\frac{\mathrm{Tr}\{[-i\,\hat{p}+m]\gamma^\rho[-i(\hat{p}-\hat{q})+m]\gamma^\delta\}}{(p^2+m^2-i\varepsilon)((p-q)^2+m^2-i\varepsilon)} \tag{5.3}
$$

Taking trace of (5.2) and keeping term of the order of θ^2, one gets

$$
\begin{aligned}
\Pi^{\rho\sigma}(q)=&\frac{-i\,e^2}{(2\pi)^4}\int d^4p\frac{\mathrm{Tr}\{[-i\,\hat{p}+m]\gamma^\rho[-i(\hat{p}-\hat{q})+m]\gamma^\delta\}}{(p^2+m^2-i\varepsilon)((p-q)^2+m^2-i\varepsilon)}\\
&\times\left[1+\frac{\theta^2}{2}((p\cdot q)^2-p^2q^2)\right]
\end{aligned} \tag{5.4}
$$

Next we would like to act as follows from the local theory.

1. Use the Feynman parameterization

$$\frac{1}{(p^2+m^2-i\varepsilon)((p-q)^2+m^2-i\varepsilon)} = \int_0^1 dx\,[(p-qx)^2+m^2+q^2x(1-x)-i\varepsilon]^{-2}$$

2. Carry out shift of the variable of integration in momentum space

$$p \to p+qx$$

3. Calculate the trace as

$$\begin{aligned}\Lambda_\theta^{\rho\sigma}(p,q) = {} & \mathrm{Tr}\,\{[-i(\hat{p}+\hat{q}x)+m]\gamma^\rho[-i(\hat{p}-q(1-x))+m]\gamma^\sigma\} \\ & \times\left[1+\frac{\theta^2}{2}((p\cdot q)^2-p^2q^2)\right] = 4[-(p+qx)^\rho(p-q(1-x))^\sigma \\ & +(p-qx)(p-q(1-x))g^{\rho\sigma}-(p+qx)^\sigma(p-q(1-x))^\rho \\ & +m^2g^{\rho\sigma}]\left[1+\frac{\theta^2}{2}((p\cdot q)^2-p^2q^2\right]\end{aligned}$$

where the factor $(p\cdot q)^2-p^2q^2$ due to noncommutativity of spacetime is invariant with respect to the shift, $p\to p+qx$. Our next step is called a Wick rotation $p^0\to -ip^4$, $d^4p\to (d^4p)_E = dp^1dp^2dp^3dp^4$ and all scalar products are evaluated using the Euclidian norm $a\cdot b = a^1b^1+a^2b^2+a^3b^3+a^4b^4$ with $q^4=-iq^0$. Also, as in the local theory, $g^{\rho\sigma}$ can be taken as either the Kronecker delta $\delta^{\rho\sigma}$, with the indices running over 1,2,3,4, or as the usual Minkowski tensor, with the indices running over 1,2,3,0. The integral

$$\Pi^{\rho\sigma}(q) = \frac{e^2}{(2\pi)^4}\int_0^1 dx\int (d^4p)_E[p^2+m^2+q^2x(1-x)]^{-2}\Lambda_\theta^{\rho\sigma}(p,q) \tag{5.5}$$

is badly divergent, which is calculated by using the dimensional regularization technique introduced in 't Hooft and Veltman (1972) based on a continuation from 4 to an arbitrary number d of spacetime dimensions.

For calculation purpose, we take into account the following formulas in d-spacetime.

1. In addition to (1.5) we have

$$\gamma^\mu\gamma_\mu = d, \qquad \gamma_\mu\gamma\nu\gamma^\mu = (2-d)\gamma_\nu \tag{5.6}$$

2. $$\mathrm{Tr}\, I = N(d), \qquad \mathrm{Tr}\, \gamma_\mu \gamma_\nu = N(d) g_{\mu\nu} \tag{5.7}$$

3. $$\mathrm{Tr}\, \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta = N(d)[g_{\mu\nu} g_{\alpha\beta} + g_{\nu\alpha} g_{\mu\beta} - g_{\mu\alpha} g_{\nu\beta}] \tag{5.8}$$

4. $$\gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma \gamma^\nu = (2-d)\gamma_\rho \gamma_\mu \gamma_\sigma + 2(\gamma_\mu \gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma \gamma_\mu) \tag{5.9}$$

5. $$\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu = (2-d)^2 \gamma_\mu \tag{5.10}$$

6. $$p^\mu p^\nu \to p^2 g^{\mu\nu}/d \tag{5.11}$$

7. $$p^\mu p^\nu p^\rho p^\sigma \to (p^2)^2 [g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}]/d(d+2) \tag{5.12}$$

Further, we use the well-known formulas:

8. $d^4 p_E \to \Omega_d k^{d-1} dk$, where $k \equiv \sqrt{p^2}$ and Ω_d is the area of a unit sphere in d dimensions

$$\Omega_d = 2\pi^{d/2}\Gamma(d/2) \tag{5.13}$$

9. There is an infinity in the one-loop contribution in NQED, arising from the limiting behavior of the Gamma function

$$\Gamma\left(2 - \frac{d}{2}\right) \to \frac{1}{(2 - d/2)} - \gamma, \tag{5.14}$$

where γ is the Euler constant, $\gamma = 0.5772157$.

10. Make use of the limiting behavior

$$\lim_{\varepsilon \to 0} a^\varepsilon \lim_{\varepsilon \to 0} e^{\varepsilon \ln a} = 1 + \varepsilon \ln a \tag{5.15}$$

where as usual we choose $\varepsilon = 2 - d/2$, for $d \to 4$.

11. To evaluate the resulting integral like

$$\int d^4 k \frac{(k^2)^n}{[k^2 + \nu^2]^m}$$

with $(k^2 + \nu^2)^m$ coming from the combined propagator denominators in Feynman diagrams for NQED, and $(k^2)^n$ coming from the propagator numerators and vertex momentum factors including $(kq)^2 - k^2 q^2$ due to noncommutativity of spacetime, we use the well-known formula

$$\int_0^\infty dk \frac{k^{l-1}}{[k^2 + \nu^2]^m} = \nu^{l-2m} \frac{\Gamma(l/2)\Gamma(m - l/2)}{2\Gamma(m)} \tag{5.16}$$

where $l = d + 2m$. In this work, we used this formula in the special cases $n = 0$, $n = 2$, $n = 4$, and $m = 2$, $m = 3$.

12. Finally, we need some properties of the Gamma-function

$$z\Gamma(z) = \Gamma(1 + z)$$

$$\psi_1(z) = \frac{d \ln \Gamma(z)}{dz} + \frac{\Gamma'(z)}{\Gamma(z)}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$$
$$\Gamma(-1+\varepsilon) = -\left[\frac{1}{\varepsilon} + 1 - \gamma + O(\varepsilon)\right]$$

or in the general case,

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) + O(\varepsilon)\right] \quad (5.17)$$

The above-listed formulas are very useful to construct gauge invariant and finite NQED in any order in θ. Here we are restricted in its second order in θ^2. After such matematical preparation, we turn to study expression (5.5) for vacuum polarization diagram.

To carry out angular averages in (5.5), we drop all terms that are odd in p, and replace the terms that have even numbers of p-factors with using (5.11) and (5.12). Also, after writing the integrand in this way as a function only of p^2, the volume element d^4_{pE} is to be replaced in accordance with (5.13). Thus, expression $\Lambda^{\rho\sigma}_{\theta}(p,q)/4$ acquires the form

$$\begin{aligned}\Lambda^{\rho\sigma}_{\theta}&(p,q)/4\\ &= \left[-\frac{2k^2}{d}g^{\rho\sigma} + 2q^{\rho}q^{\sigma}x(1-x) + (k^2 - q^2x(1-x))g^{\rho\sigma} + m^2g^{\rho\sigma}\right]\\ &\quad + \frac{\theta^2}{2}\left\{\frac{1-d}{d}q^2\cdot k^2\tau^{\rho\sigma} + \frac{2}{d}\frac{1+d}{d+2}\cdot q^2\cdot k^4g^{\rho\sigma} - \frac{2k^4}{d(d+2)}\cdot 2q^{\rho}q^{\sigma}\right\}\end{aligned} \quad (5.18)$$

where ($k \equiv p$),

$$\begin{aligned}k^2\tau^{\rho\sigma} &= k^2[2q^{\rho}q^{\sigma}x(1-x) + m^2g^{\rho\sigma}] - k^2q^2x(1-x)g^{\rho\sigma} + k^4g^{\rho\sigma}\\ &= k^2[2q^{\rho}q^{\sigma}x(1-x) - q^2x(1-x)g^{\rho\sigma} + m^2g^{\rho\sigma}] + k^4g^{\rho\sigma}\end{aligned}$$

We now use integrals of the type of (5.16):

1) $\int_0^{\infty} dk\, k^{d-1}[k^2+\nu^2]^{-2} = \frac{1}{2}(\nu^2)^{\frac{d}{2}-2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)$
2) $\int_0^{\infty} dk\, k^{(d+2)-1}[k^2+\nu^2]^{-2} = \frac{1}{2}(\nu^2)^{\frac{d}{2}-1}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$
3) $\int_0^{\infty} dk\, k^{(d+4)-1}[k^2+\nu^2]^{-2} = \frac{1}{2}(\nu^2)^{\frac{d}{2}}\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)$

Then, expression (5.5) consists of two parts

$$\prod^{\rho\sigma}(q) = \prod^{\rho\sigma}_{\text{local}}(q) = \prod^{\rho\sigma}_{1\theta}(q) \quad (5.19)$$

where

$$\Pi^{\rho\sigma}_{\text{local}}(q) = \frac{4\,e^2\Omega_d}{(2\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)[q^\rho q^\sigma - q^2 g^{\rho\sigma}] \times \int_0^1 dx\cdot x(1-x)[m^2+q^2x(1-x)]^{\frac{d}{2}-2} \tag{5.20}$$

is the usual local quantum electrodynamical result and

$$\begin{aligned}\Pi^{\rho\sigma}_{1\theta}(q) = \frac{\theta^2}{2}\frac{2\,e^2\Omega_d}{(2\pi)^4}\int_0^1 dx\Bigg\{&\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)[m^2+q^2x(1-x)]^{\frac{d}{2}}\\ &\times\left[\frac{1-d}{d}q^2g^{\rho\sigma}+\frac{2}{2}\frac{1+d}{d+2}\cdot q^2g^{\rho\sigma}-\frac{4}{d(d+2)}\cdot q^\rho q^\sigma\right]\\ &+\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)[m^2+q^2x(1-x)]^{\frac{d}{2}-1}\cdot\frac{1-d}{d}q^2\\ &\times[2q^\rho q^\sigma x(1-x)-q^2x(1-x)g^{\rho\sigma}+m^2g^{\rho\sigma}]\Bigg\}\end{aligned} \tag{5.21}$$

For the last formula, we use the following transformations:

1) $$\frac{1-d}{d}+\frac{2(1+d)}{d(d+2)}=\frac{4}{d(d+2)}-\frac{d(d-1)}{(d+2)d}$$

2) $$\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\left(-\frac{d}{d+2}\right)=\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$$

and therefore, one can combine first terms in the first expression of (5.21) with its second whole expression to give

3) $$\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1-d}{d}\right)[m^2+q^2x(1-x)]^{\frac{d}{2}-1} \times[2q^pq^\sigma q^2x(1-x)-2q^4x(1-x)g^{\rho\sigma}]$$

Thus, expression (5.21) takes the form

$$\begin{aligned}\Pi^{\rho\sigma}_{1\theta}(q) = \frac{2\,e^2\Omega_d}{(2\pi)^4}\cdot\frac{\theta^2}{2}\int_0^1 dx\,\Bigg\{&\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\frac{4}{d(d+2)}[m^2+q^2x(1-x)]^{\frac{d}{2}}\\ &\times(q^2g^{\rho\sigma}-q^pq^\sigma)-\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1-d}{d}\right)\\ &\times\left[m^2+q^2x(1-x)\right]^{\frac{d}{2}-1}\cdot 2q^2x(1-x)(q^2g^{\rho\sigma}-q^pq^\sigma)\Bigg\}\end{aligned} \tag{5.22}$$

After little algebra of the Gamma functions

$$\Gamma\left(2+\frac{d}{2}\right)=\frac{d+2}{2}\Gamma\left(1+\frac{d}{2}\right),\quad \Gamma\left(1-\frac{d}{2}\right)=-\frac{d}{2}\Gamma\left(-\frac{d}{2}\right)$$

and

$$-\frac{1}{d}\Gamma\left(1-\frac{d}{2}\right)=\frac{1}{2}\Gamma\left(-\frac{d}{2}\right)$$

one gets

$$\begin{aligned}\Pi_{1\theta}^{\rho\sigma}(q)=&\frac{2\,e^2\Omega_d}{(2\pi)^4}\cdot\frac{\theta^2}{2}(q^2q^{\rho\sigma}-q^pq^\sigma)\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\\ &\times\int_0^1 dx\left[\frac{2}{d}[m^2+q^2x(1-x)]^{\frac{d}{2}}+(1-d)q^2x(1-x)\right.\\ &\left.\times\,(m^2+q^2x(1-x))^{\frac{d}{2}-1}\right]\end{aligned}\tag{5.23}$$

Thus, we find vacuum polarization in NQED

$$\begin{aligned}\Pi^{\rho\sigma}(q)=&\frac{4\,e^2\Omega_d}{(2\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)(q^pq^\sigma-q^2g^{\rho\sigma})\\ &\times\int_0^1 dx[m^2+q^2x(1-x)]^{\frac{d}{2}-2}\left\{x(1-x)+\frac{\theta^2}{2}\cdot\frac{1}{2-d}\right.\\ &\left.\times\,(m^2+q^2x(1-x))\left[\frac{d}{2}(m^2+q^2x(1-x))+(1-d)q^2x(1-x)\right]\right\}\end{aligned}\tag{5.24}$$

We note the very remarkable result that this contribution satisfies relation

$$q_\rho\Pi^{\rho\sigma}(q)=0\tag{5.25}$$

that is the basis of the conservation and neutrality of the electric current in NQED in which dimensional regularization gives also this result of the conservation of current that does not depend on the dimensionality of spacetime.

Owing to (5.14) the Gamma function $\Gamma(2-\frac{d}{2})$ in (5.25) has singularity at the limit $d-4$. Moreover, as shown in Section 4, there is another term that must be added to $\Pi^{\rho\sigma}(q)$, arising from the term $-\frac{1}{4}(Z_3-1)F_\theta^{\mu\nu}\star F_{\mu\nu}^\theta$ in the interaction Lagrangian. This term has a structure like Eq. (5.24)

$$\Pi_{\mathcal{L}_2}^{\rho\sigma}(q)=-(Z_3-1)(q^2g^{\rho\sigma}-q^pq^\sigma)\tag{5.26}$$

so as to order e^2, the full $\Pi_f^{\rho\sigma}$ has the form

$$\Pi_f^{\rho\sigma}=(q^2g^{\rho\sigma}-q^pq^\sigma)\Pi_\theta(q^2)\tag{5.27}$$

with

$$\Pi_\theta(q^2) = -\frac{4\,e^2\Omega_d}{(2\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx[m^2+q^2x(1-x)]^{\frac{d}{2}-2}$$
$$\times\left\{x(1-x)+\frac{\theta^2}{2d(2-d)}(m^2+q^2x(1-x))\right.$$
$$\left.\times\,[2m^2+q^2x(1-x)(2+d-d^2)]\right\}-(Z_3-1) \qquad (5.28)$$

As in the local QED, the definition of the noncomutative renormalized electromagnetic field requires that $\Pi_\theta(0)=0$. Therefore, to order e^2,

$$Z_3 = 1-\frac{4\,e^2\Omega_d}{(2\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)(m^2)^{\frac{d}{2}-2}\int_0^1 dx\left[x(1-x)+\frac{\theta^2}{d(2-d)}\cdot m^4\right]$$
$$\times\left[(m^2+q^2x(1-x))^{\frac{d}{2}-2}-(m^2)^{\frac{d}{2}-2}\right] \qquad (5.29)$$

Now we can remove the regulatization allowing d to approach its physical value $d=4$. There is an infinity in the one-loop contribution, arising from the limiting behavior of the Gamma function (5.14). According to the local QED a finite part of $\Pi_\theta(q^2)$. is extracted from the mathematical prescription

$$\Pi_\theta^f(q^2)=\Pi_\theta(q^2)-\Pi_\theta(0)-\left.\frac{\partial\Pi_\theta(q^2)}{\partial q^2}\right|_{q^2=0}\cdot q^2-\frac{1}{2}\left.\frac{\partial\Pi_\theta(q^2)}{\partial q^4}\right|_{q^2=0}\cdot q^4 \quad (5.30)$$

A straightforward calculation gives

1) $$\Pi_\theta(0)=I\cdot(m^2)^{\frac{d}{2}-2}(x(1-x)+\tilde\theta^2\cdot 2m^4),$$

2) $$\left.\frac{\partial\Pi_\theta(q^2)}{\partial q^2}\right|_{q^2=0}\cdot q^2$$
$$=I\left\{\left(\frac{d}{2}-2\right)(m^2)^{\frac{d}{2}-3}\cdot x(1-x)+(x(1-x)+\tilde\theta^2\cdot 2m^4)+(m^2)^{\frac{d}{2}-2}\right.$$
$$\left.\times\,[\tilde\theta^2\cdot x(1-x)2m^2+\tilde\theta^2\cdot m^2(2+d-d^2)x(1-x)]\right\}q^2, \qquad (5.31)$$

and

3) $$\left.\frac{\partial\Pi_\theta(q^2)}{\partial q^4}\right|_{q^2=0}\cdot q^4=I\left\{\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)(m^2)^{\frac{d}{2}-4}\cdot x^2(1-x)^2\right.$$
$$\times\,(x(1-x)+\tilde\theta^2\cdot 2m^4)+2\left(\frac{d}{2}-2\right)(m^2)^{\frac{d}{2}-3}\cdot x^2(1-x)^2m^2\tilde\theta^2$$
$$\left.\times\,(4+d-d^2)+2(m^2)^{\frac{d}{2}-2}\tilde\theta^2\cdot x^2(1-x)^2(2+d-d^2)\right\}\cdot q^4 \quad (5.32)$$

where

$$\tilde{\theta}^2 = \frac{\theta^2}{2-d}\cdot\frac{1}{2d}, \quad I = -\frac{4\,e^2\Omega_d}{(2\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx$$

The poles at $d = 4$ obviously cancel in $\Pi_\theta(q^2)$, because for $d = 4$ both $(m^2 + q^2x(1-x))^{\frac{d}{2}-2}$ and $(m^2)^{\frac{d}{2}-2}$ have the same limit, unity (see formula (5.15)). For the same reason, the term $-\gamma$ in $\Gamma(2-d/2)$ cancels in the total $\Pi(q^2)$, although γ it does make a finite contribution to $Z_3 - 1$. There are other finite contributions to $Z_3 - 1$, that arise from the product of the pole in $\Gamma(2-d/2)$ with the linear terms in the expansion of $\Omega_d\Gamma(d/2)$ around $d = 4$, but these also cancel in the total $\Pi_\theta(q^2)$.

The only terms that do contribute to $\Pi_\theta(q^2)$ in the limit $d \to 4$ are those arising from the product of the pole in $\Gamma(2-d/2)$ with the linear terms in the expansion of $(m^2 + q^2x(1-x))^{\frac{d}{2}-2}$ and $(m^2)^{\frac{d}{2}-2}$ in powers of $d-4$:

$$(m^2 + q^2x(1-x))^{\frac{d}{2}-2} - (m^2)^{\frac{d}{2}-2} \to \left(\frac{d}{2}-2\right)\ln\left(1+\frac{q^2x(1-x)}{m^2}\right)$$

due to the formula (5.15) and are also those arising from the product $\Gamma(2-d/2)\cdot(\frac{d}{2}-2)$ in (5.30).

Finally, all these simpler calculations give

$$\Pi_\theta(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\left\{\left[x(1-x) - \frac{\theta^2}{16}(2m^4 - 8m^2q^2x(1-x) - 10x^2(1-x)^2q^4)\right]\right.$$
$$\left.\ln\left(1+\frac{q^2x(1-x)}{m^2}\right) + \frac{\theta^2}{16}(2m^2q^2x(1-x) - 9x^2(1-x)^2q^4)\right\} \tag{5.33}$$

The physical importance of the vacuum polarization in NQED can be explored by considering its effect on the scattering of two charged particles of spin $\frac{1}{2}$. The Feynman diagrams of Fig. 3 make contributions to the scattering S_θ-matrix element of the form

$$S^a_\theta(1,2\to 1',2') = e^{-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_2^\rho+p_1^\rho)p_1'^\sigma}(2\pi)^{-6}\delta^4(p_1'+p_2'-p_1-p_2)$$
$$\times[e_1(2\pi)^4\overline{u}(p_1')\gamma^\mu u(p_1)]\left[-i(2\pi)^{-4}\frac{1}{q^2}\right][e_2(2\pi)^4\overline{u}(p_2')\gamma_\mu u(p_2)],$$

$$S^b_\theta(1,2\to 1',2') = e^{-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_2^\rho+p_1^\rho)p_1'^\sigma}(2\pi)^{-6}\delta^4(p_1'+p_2'-p_1-p_2)$$
$$\times[e_1(2\pi)^4\overline{u}(p_1')\gamma^\mu u(p_1)]\left[-i(2\pi)^{-4}\frac{1}{q^2}\right]^2$$
$$\times[i(2\pi)^4(q^2g^{\mu\nu}+g_\mu g_\nu)\Pi_\theta(q^2)][e_2(2\pi)^4\overline{u}(p_2')\gamma^\nu u(p_2)]$$

where the factor $\exp[-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_1+p_2)^\rho p_1'^\sigma]$ is arising form the $\star$-product:

$$e^{-ip_1x}\star e^{ip_2x}\star e^{-iq(x-y)}\star e^{-ip_1'y}\star e^{ip_2'y}$$

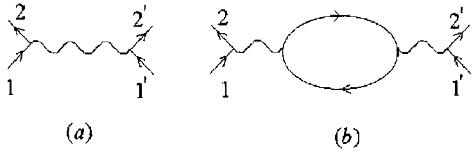


Fig. 3. Two diagrams for the scattering of charged particles in NQED.

In the momentum space it takes the form:

$$\exp\left\{\frac{1}{2}\theta\sigma_{\mu\nu}p_1^{\mu}p_2^{\nu}+\frac{1}{2}\theta\sigma_{\rho\sigma}p_2^{\rho}(p_2'-p_2)^{\sigma}+\frac{1}{2}\theta\sigma_{\alpha\beta}p_1'^{\alpha}(p_1-p_1')^{\beta}\right.$$
$$\left.+\frac{1}{2}\theta\sigma_{\lambda\chi}p_1'^{\lambda}p_2'^{\chi}\right\}=\exp\left[-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_1+p_2)^{\rho}p_1'^{\sigma}\right]$$

and e_1 and e_2 are the charges of the two particles being scattered: $\Pi_\theta(q^2)$ is calculated using for e in Eq. (5.33) the magnitude of the charge of the particle circulating in the loop in Fig. 3; and q^μ is the momentum transfer $q=p_1-p_1'=p_2'-p_2$. Using the conservation property $q_\mu\overline{u}(p_1')\gamma^\mu u(p_1)=0$ the two diagrams together yield an S_θ-matrix element:

$$S_\theta^{a+b}(1,2\to 1',2')=\frac{-i\,e_1\,e_2}{4\pi^2q^2}\,e^{-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_1+p_2)^{\rho}p_1'^{\sigma}}[1+\Pi_\theta(q^2)]$$
$$\times\delta^4(p_1'+p_2'-p_1-p_2)[\overline{u}(p_1')\gamma^\mu u(p_1)][\overline{u}(p_2')\gamma_\mu u(p_2)] \tag{5.34}$$

In the nonrelativistic limit, $\overline{u}(p_1')\gamma^0u(p_1)\sim i\delta_{\sigma_1'\sigma_1}$ while $\overline{u}(p_1')\gamma^iu(p_1)\simeq 0$, and likewise for particle 2. In this limit q^0 is also negligible compared with $|\vec{q}|$. Equation (5.34) in this limit becomes

$$S_\theta^{a+b}(1,2\to 1',2')=\frac{-i\,e_1\,e_2}{4\pi^2q^2}\,e^{-\frac{1}{2}\theta\sigma_{\rho\sigma}(p_1+p_2)^{\rho}p_1'^{\sigma}}[1+\Pi_\theta(q^2)]$$
$$\times\delta^{(4)}(p_1'+p_2'-p_1-p_2)\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2} \tag{5.35}$$

This expression may be compared with the S-marix in the Born approximation due to a local spin-independent central potential $V(r)$:

$$S_\theta^{\text{Born}}(1,2\to 1',2')=-2\pi i\delta(E_1'+E_2'-E_1-E_2)\,e^{\frac{1}{2}\theta\sigma_{00}(E_1+E_2)E_1'}$$
$$\times T_{\text{Born}}(1,2\to 1',2') \tag{5.36}$$

$$T_{\text{Born}}(1,2\to 1',2')=\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\int d^3x_1\int d^3x_2V(|\mathbf{x}_1-\mathbf{x}_2|)$$
$$\times(2\pi)^{-6}\,e^{-i\mathbf{P}_2'\mathbf{x}_1}\star e^{-i\mathbf{P}_1'\mathbf{x}_2}\star e^{i\mathbf{P}_1\mathbf{x}_1}\star e^{i\mathbf{P}_2\mathbf{x}_2} \tag{5.37}$$

Setting $\mathbf{x}_1 = \mathbf{x}_2 + \mathbf{r}$, this yields

$$T_{\text{Born}} \approx \frac{-i}{4\pi^2}\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\, e^{-\frac{1}{2}\theta\sigma_{ij}p_{1i}'\varepsilon_j}\delta(p_1' + p_2' - p_1 - p_2)$$
$$\times \int d^3r\, V(r) \star e^{-i\mathbf{qr}}\left[1 + \frac{\theta^2}{2}[(\mathbf{p}_1 \cdot (\mathbf{q} + \vec{\varepsilon}))^2 - \mathbf{p}_1^2(\mathbf{q} + \vec{\varepsilon})^2\right] \quad (5.38)$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_1' = \mathbf{p}_2' - \mathbf{p}_2$ and $\vec{\varepsilon} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$ are the momentum transfer and the total energy of the system. Comparing this with Eq. (5.34) shows that in the nonrelativistic limit the diagrams of Fig. 3 give the same S_θ-matrix element as a potential $V(r)$ such that

$$\int d^3r\, V(r) \star e^{-i\mathbf{qr}} \approx e_1\, e_2 \frac{1 + \Pi_\theta(\mathbf{q}^2)}{\mathbf{q}^2}$$

or, inverting the Fourier transform

$$V(r) = \frac{e_1\, e_2}{(2\pi)^3}\int d^3q\, e^{i\mathbf{qr}}\left[\frac{1 + \Pi_\theta(\mathbf{q}^2)}{\mathbf{q}^2}\right] \quad (5.39)$$

Equation (5.39) is to first order in the radiative correction the same potential energy that would be produced by the electrostatic interaction of two extended charges distribution $e_1\rho_\theta(\mathbf{x})$ and $e_2\rho_\theta(\mathbf{y})$ at a distance r:

$$V(|\mathbf{r}|) = e_1\, e_2 \int d^3x \int d^3y \frac{\rho_\theta(\mathbf{x}) \star \rho_\theta(\mathbf{y})}{4\pi\, |\mathbf{x} - \mathbf{y} + \mathbf{r}|} \quad (5.40)$$

where

$$\rho_\theta(\mathbf{x}) = \left[1 - \frac{e^2\theta^2}{64\pi^2(2\pi)^3}\left(\frac{1}{3}m^2\vec{\nabla}^2 + \frac{3}{10}\vec{\nabla}^4\right)\right]\delta^3(\mathbf{r})$$
$$+ \frac{1}{2(2\pi)^3}\int d^3q\, \Pi_{1\theta}(\mathbf{q}^2)\, e^{i\mathbf{qr}} \quad (5.41)$$

Here $\Pi_{1\theta}(\mathbf{q}^2)$ is given by the term with $\ln(1 + \frac{q^2(1-x)}{m^2})$ in (5.33). Note that

$$\int d^3r\, \rho_\theta(\mathbf{r}) = 1 + \frac{1}{2}\Pi_\theta(0) = 1 \quad (5.42)$$

so the total charges of particles 1 and 2, as determined from the long-range part of the Coulomb potential, are the same constants e_1 and e_2 that govern the interactions of the renormalized electromagnetic field in NQED.

As in the local QED, for $|\mathbf{r}| \neq 0$ the integral (5.41) can be carried out by a straightforward contour integration:

$$\rho_\theta(\mathbf{r}) = -\frac{e^2}{8\pi^3 r^3}\int_0^1 dx \cdot x(1 - x)\left\{1 - \frac{\theta^2}{8}\left[\frac{m^4}{x(1 - x)} + 4m^2\vec{\nabla}^2\right.\right.$$

$$-5x(1-x)\vec{\nabla}^4\Big]\Big\}\left(1+\frac{mr}{\sqrt{x(1-x)}}\right)\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right)$$

On the other hand, the integral of $\rho_\theta(\mathbf{r})$ over all $\mathbf{r}$ equals $+1$. Therefore, $\rho_\theta(\mathbf{r})$ must contain a term $(1+N_\theta)\delta^3(\mathbf{r})$ that is singular at $r=0$, with N_θ chosen to satisfy Eq. (5.42):

$$N_\theta = \frac{e^2\theta^2}{64\pi^2(2\pi)^3}\left(-\frac{1}{3}m^2\vec{\nabla}^2-\frac{3}{10}\vec{\nabla}^4\right)+\frac{e^2}{8\pi^3}\int\frac{d^3r}{r^3}$$
$$\times\int_0^1 dx\cdot x(1-x)\left\{1-\frac{\theta^2}{8}\left[\frac{m^4}{x(1-x)}+4m^2\vec{\nabla}^2-5x(1-x)\vec{\nabla}^4\right]\right\}$$
$$\left(1+\frac{mr}{\sqrt{x(1-x)}}\right)\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) \tag{5.43}$$

The complete expression for the charge distribution function is then

$$\rho_\theta(\mathbf{r}) = (1+N_\theta)\delta^3(\mathbf{r})-\frac{e^2}{8\pi^3r^3}\int_0^1 dx\cdot x(1-x)\left\{1-\frac{\theta^2}{8}\left[\frac{m^4}{x(1-x)}+4m^2\vec{\nabla}^2\right.\right.$$
$$\left.\left.-5x(1-x)\vec{\nabla}^4\right]\right\}\left(1+\frac{mr}{\sqrt{x(1-x)}}\right)\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) \tag{5.44}$$

The physical meaning of this result is that a bare point charge attracts of charge of opposite sign out of the vacuum, repelling their antiparticles to infinity, so that the bare charge is partially shielded, yielding a renormalized charge smaller by a factor $1/(1+N_\theta)$.

The vacuum polarization effect of Feynman graph (b) in Fig. 3 is to shift the energy of an atomic state with wave function $\psi(\mathbf{r})$ by

$$\Delta E_\theta = \int d^3r\,\Delta V(r)\star[\psi^*(\mathbf{r})\star\psi(\mathbf{r})]$$
$$=\frac{1}{(2\pi)^9}\int d^3r\int d^3q\int d^3Q\int d^3k\,e^{i\mathbf{qr}}\star e^{-i\mathbf{kr}}\star e^{i\mathbf{Qr}}$$
$$\times\Delta\tilde{V}(\mathbf{q})\tilde{\psi}^*(\mathbf{k})\tilde{\psi}(\mathbf{Q}) = \frac{1}{(2\pi)^9}\int d^3r\int d^3q\int d^3Q\int d^3k$$
$$\times\exp\left[\frac{1}{2}\theta\sigma_{\mu\nu}q^\mu k^\nu+\frac{1}{2}\theta\sigma_{\rho\sigma}k^\rho Q^\sigma\right]\Delta\tilde{V}(\mathbf{q})\tilde{\psi}^*(\mathbf{k})\tilde{\psi}(\mathbf{Q})e^{i\mathbf{qr}-i\mathbf{kr}+i\mathbf{Qr}} \tag{5.45}$$

Here

$$\lambda = \exp\left[\frac{1}{2}\theta\sigma_{ij}q^i k^j + \frac{1}{2}\theta\sigma_{nm}k^n Q^m\right]$$
$$= \exp\left[\frac{1}{2}\theta\sigma_{ij}(q^i + Q^i)k^j\right], \qquad \sigma_{ij} = \sigma_i\sigma_j - \sigma_j\sigma_i$$

(σ_{ij} are the Pauli matrices)

and

$$\langle\lambda\rangle = \frac{1}{N(d)}\mathrm{Tr}\lambda = \cosh\left(\theta\sqrt{((\mathbf{q}+\mathbf{Q})\cdot\mathbf{k})^2 - \mathbf{k}^2\cdot(\mathbf{q}+\mathbf{Q})^2}\right)$$

We find

$$\Delta E_\theta = \cosh\left(\theta\sqrt{(\vec{\nabla}_x\cdot\vec{\nabla}_y)^2 - \Delta_x\cdot\Delta_y}\right)\int d^3x\,\Delta V(\mathbf{x})\psi^*(\mathbf{y})\psi(\mathbf{x})|_{x=y} \quad (5.46)$$

Here $\Delta V(\mathbf{x})$ is the perturbation in the potential (5.39):

$$\Delta V(\mathbf{x}) = \frac{e_1\,e_2}{(2\pi)^3}\int d^3q\, e^{i\mathbf{qr}}\left[\frac{\Pi_\theta(\mathbf{q}^2)}{\mathbf{q}^2}\right] \quad (5.47)$$

We know that the effect of the vacuum polarization is very much larger for orbital angular momentum $l = 0$ than for its higher values. For $l = 0$ the wave function is approximately equal to the constant for r less than or of the order of m^{-1}, so Eq. (5.46) becomes

$$\Delta E_\theta = |\psi(0)|^2\int d^3r\,\Delta V(\mathbf{r}) \quad (5.48)$$

since $\cosh 0 = 1$. Using Eqs. (5.47) and (5.33), the integral of the shift in the potential (for $e_1\,e_2 = -Ze^2$) is

$$\int d^3r\; V(\mathbf{r}) = -Ze^2\Pi'_\theta(0) \quad (5.49)$$

Direct calculation of Eq. (5.33) gives

$$\Pi'_\theta(0) = \frac{e^2}{2\pi^2}\int_0^1 dx\left[\frac{x^2(1-x)^2}{m^2} - \frac{\theta^2 m^2 x(1-x)}{8} + \frac{\theta^2 m^2}{8}x(1-x)\right]$$

and therefore,

$$-Z\,e^2\Pi'_\theta(0) = -\frac{4}{15}Z\frac{\alpha^2}{m^2}, \quad \alpha = \frac{e^2}{4\pi}$$

We see that an immediate contribution to the energy shift due to noncommutativity of spacetime is zero at least up to order of θ^4. Therefore, in states of hydrogenic atoms with $l = 0$ and principal quantum number n the wave function at the origin is

$$\psi(0) = \frac{2}{\sqrt{4\pi}} \left(\frac{Z\alpha m}{n} \right)^{3/2} \tag{5.50}$$

so the energy shift (5.48) is almost equal to the local one

$$\Delta E = -\frac{4}{15} \frac{Z^4 \alpha^5 m}{\pi n^3} + O(\theta^4) \tag{5.51}$$

Finally, notice that although vacuum polarization contributes only a small part of the radiative corrections in ordinary atoms, it dominates the radiative corrections in muonic atoms, in which a muon takes the place of the orbiting electron.

6. ELECTRON SELF-ENERGY IN NONCOMMUTATIVE QUANTUM ELECTRODYNAMICS

The complete electron propagator in NQED is given by the sum

$$\begin{aligned} \left[-i(2\pi)^{-4} S'_\theta(p)\right] &= [-i(2\pi)^{-4} S(p)] \\ &+ [-i(2\pi)^{-4} S(p)][i(2\pi)^4 \Sigma_\theta(p)][-i(2\pi)^{-4} S(p)] + \cdots \end{aligned}$$

where

$$S(p) = \frac{-i\hat{p} + m_e}{p^2 + m_e^2 - i\varepsilon}$$

The sum is trivial, and gives

$$S'_\theta(p) = [i\hat{p} + m_e - \Sigma_\theta - i\varepsilon]^{-1} \tag{6.1}$$

In lowest order there is a one-loop contribution to Σ_θ, given by Fig. 4:

$$-i : \bar{\psi}(x) \star \Sigma_\theta(x-y) \star \psi(y) :$$

where

$$\Sigma_\theta(x-y) = -i\, e^2 \Delta(x-y) \star \gamma^\mu S(x-y) \gamma_\mu$$

The $\star$-product:

$$e^{-iqx} \star e^{-ip(x-y)} \star e^{-i(p-q)(y-x)} \star e^{iqy}$$

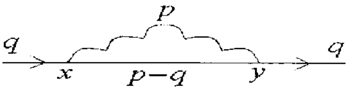


Fig. 4. The one-loop diagram for the electron self-energy function in NQED.

leads to the factor

$$e^{-\frac{1}{2}\theta\sigma_{\mu\nu}q^{\mu}p^{\nu}} \tag{6.2}$$

in the momentum space.

Similar to the vacuum polarization, the electron self-energy function has the form

$$\Sigma_\theta(q) = \frac{i\,e^2}{(2\pi)^4}\int d^4p\,\frac{1}{p^2 - i\varepsilon}\left[\frac{\gamma^\rho(-i\hat{q} + i\hat{p} + m)\gamma_\rho}{(q-p)^2 + m^2 - i\varepsilon}\right.$$
$$\left.\times\left\{1 + \frac{\theta^2}{2}((p.q)^2 - p^2q^2)\right\}\right] \tag{6.3}$$

here and below we have used notation $m = m_e$.

Making use of the Feynman parameterization formula, invariance property of the factor $(pq)^2 - p^2q^2$ with respect to the shift $p \to p + qx$ and the formula (5.6), one gets

$$\Sigma_\theta(q) = \frac{i\,e^2}{(2\pi)^4}\int_0^1 dx\int d^4p\left[(p-qx)^2 + q^2x(1-x) + m^2x - i\varepsilon\right]^{-2}$$
$$\times[-i(\hat{q} - \hat{p})(2-d) + md]\left[1 + \frac{\theta^2}{2}((p.q)^2 - p^2q^2)\right]$$

Going to the Wick rotation and using the d-dimensional regularization procedure as before, we obtain

$$\Sigma_\theta(q) = -\frac{e^2\pi^{d/2}}{2(\pi)^4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2 - \frac{d}{2}\right)\int_0^1 dx[-i(2-d)(1-x)\hat{q} + dm]$$
$$\times[q^2x(1-x) + m^2x]^{\frac{d}{2}-2}\left[1 + \frac{\theta^2}{x}\cdot\frac{1-d}{2-d}q^2(q^2x(1-x) + m^2x)\right] \tag{6.4}$$

The interaction (4.7) also contributes a renormalization counterterm $-(Z_2 - 1)(i\hat{q} + m) + Z_2\delta m$ in $\Sigma_\theta(p)$ with Z_2 and δm determined by the condition that the complete propagator $S'_\theta(p)$ regarded as a function of $i\hat{q}$ should have a pole at $i\hat{q} = -m$ with residue unity.

To remove the regularization, allowing d to got to its limit $d \to 4$, we act as follows. We calculate the quantity $\Sigma_\theta(q)$ and its derivatives

$$\frac{\partial\Sigma}{\partial i\hat{q}},\quad \frac{\partial^2\Sigma}{\partial(i\hat{q})^2},\quad \frac{\partial^3\Sigma}{\partial(i\hat{q})^3}\cdots\quad \frac{\partial^5\Sigma}{\partial(i\hat{q})^5}$$

at the point $i\hat{q} = -m$, and use the Taylor series:

$$\Sigma_\theta(p) - \Sigma_\theta(p)|_{i\hat{q}=-m} + i\frac{\partial \Sigma_\theta}{\partial \hat{q}}|_{i\hat{q}=-m}(i\hat{q}+m) - \frac{1}{2!}\frac{\partial^2 \Sigma_\theta}{\partial (i\hat{q})^2}|_{i\hat{q}=-m}(i\hat{q}+m)^2$$
$$- \frac{1}{3!}\frac{\partial^3 \Sigma_\theta}{\partial (i\hat{q})^3}|_{i\hat{q}=-m}(i\hat{q}+m)^3 - \frac{1}{4!}\frac{\partial^4 \Sigma_\theta}{\partial (i\hat{q})^4}|_{i\hat{q}=-m}(i\hat{q}+m)^4$$
$$- \frac{1}{5!}\frac{\partial^5 \Sigma_\theta}{\partial (i\hat{q})^5}|_{i\hat{q}=-m}(i\hat{q}+m)^5 \tag{6.5}$$

Since, one can write

$$(i\hat{q}+m)^2 = (m+i\hat{q})[2m] + m\left[-m\frac{q^2}{m}\right],$$
$$(i\hat{q}+m)^3 = (m+i\hat{q})[-q^2+3m^2] + m[-2m^2-2q^2],$$
$$(i\hat{q}+m)^4 = (m+i\hat{q})[4m^3-4mq^2] + \left[\frac{(m^2+q^2)^2}{m} - 4m(q^2+m^2)\right],$$
$$(i\hat{q}+m)^5 = (m+i\hat{q})[(m^2+q^2)^2 - 4m^2(q^2+m^2) + 2m(4m^3-4mq^2)]$$
$$+ m[(4m^2-4q^2)(-q^2-m^2)] \tag{6.6}$$

then a renormalization counterterm in $\Sigma_\theta(p)$ has the general form

$$-\left(Z_2^\theta - 1\right)(i\hat{q}+m) + Z_2^\theta \cdot m$$

as should expected. It turns out that the poles at $d = 4$ cancel in the definition of (6.5). First, we separate the local value of (6.5):

$$\Sigma_{\text{local}}(q) = -\frac{2e^2\pi^2}{(2\pi)^4}\int_0^1 dx\left\{2\cdot\frac{1-x^2}{x}\cdot(i\hat{q}+m)\right.$$
$$\left. + [i\hat{q}(1-x)+2m]\ln\left(\frac{m^2x^2}{q^2x(1-x)+m^2x}\right)\right\} \tag{6.7}$$

Here there is still a divergence from the behavior of the first term as $x \to 0$, which can be traced to the singular behavior of the integral over the photon momentum p in Eq. (6.3) at $p^2 = 0$, when we take q^2 at the point $q^2 = -m^2$, where we evaluated $Z_2 - 1$. Such infrared divergences have common root as in the local QED.

To show explicit cancellation of poles at the limit $d \to 4$, we calculate whole expression (6.5). Thus, terms proportional to $-\theta^2(1-d)/2(2-d)$ are as follows.

1. $$\frac{\partial^2 \Sigma_\theta^1}{\partial (\hat{q})^2} = \hat{q}q^2x(1-x)L\cdot R\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+ 2RL^{\frac{d}{2}-2}[2L\hat{q} + 2\hat{q}q^2x(1-x)]$$

$$+4q^4x^2(1-x)^2L\cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+2q^2\cdot Lx(1-x)\cdot D\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+4q^2x(1-x)D\cdot\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}[2L+2q^2x(1-x)]$$
$$+D\cdot L^{\frac{d}{2}-2}[2L+10q^2x(1-x)] \tag{6.8}$$

where we have denoted $L=q^2x(1-x)+m^2x$, $D=[-i(2-d)(1-x)\hat{q}+d\cdot m]$ and $R=-i(2-d)(1-x)$. Similar expressions hold for other terms.

2. $$\frac{\partial^3\Sigma^1_\theta}{\partial(\hat{q})^3}=12q^4x^2(1-x)^2\cdot L\cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+6q^2x(1-x)L\cdot R\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}+12q^2x(1-x)R\left(\frac{d}{2}-2\right)$$
$$+L^{\frac{d}{2}-3}[2L+2q^2x(1-x)]+3RL^{\frac{d}{2}-2}[2L+10q^2x)(1-x)]$$
$$\times 10\hat{q}x^2(1-x)^2\cdot q^2L\cdot D\cdot\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+24\cdot\hat{q}q^2x^2(1-x)^2\cdot L\cdot D\cdot\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+6\hat{q}^2x(1-x)D\cdot\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}\cdot[2L+2q^2x(1-x)]$$
$$+6\hat{q}x(1-x)D\cdot\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}\cdot[2L+10q^2x(1-x)]$$
$$+24\hat{q}x(1-x)D\cdot L^{\frac{d}{2}-2} \tag{6.9}$$

For completeness, we write two more lengthy terms.

3. $$\frac{\partial^4\Sigma^1_\theta}{\partial(\hat{q})^4}=96\hat{q}\cdot q^2x^2(1-x)^2\cdot L\cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+48\hat{q}x(1-x)R\cdot\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}\cdot[L+q^2x(1-x)]$$
$$+24\cdot\hat{q}x(1-x)R\cdot\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}\cdot[2L+10q^2x(1-x)]$$
$$+46\hat{q}q^2x^2(1-x)^2L\cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$

$$+10q^2Lx^2(1-x)^2 \cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+92q^2Lx^2(1-x)^2 \cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+36q^2Lx^2(1-x)^2 \cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+12x(1-x) \cdot D\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}[2L+10q^2x(1-x)]$$
$$+192q^2x^2(1-x)^2 \cdot D\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+96\hat{q}x(1-x) \cdot RL^{\frac{d}{2}-2}+24x(1-x)DL^{\frac{d}{2}-2}, \tag{6.10}$$

and

4. $$\frac{\partial^5 \Sigma_\theta^1}{\partial(\hat{q})^5} = 48q^2x^2(1-x)^2 \cdot 12 \cdot R\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+472q^2x^2(1-x)^2 \cdot L \cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+56q^2x^2(1-x)^2L \cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+112\hat{q}x^2(1-x)^2 \cdot L \cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+212\hat{q}x^2(1-x)^2 \cdot L \cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+228q^2x^2(1-x)^2 \cdot L \cdot R\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)L^{\frac{d}{2}-4}$$
$$+120x(1-x)^2 \cdot R\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}[L+5q^2x(1-x)]$$
$$+768q^2x^2(1-x)^2 \cdot R\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+384\hat{q}x^2(1-x)^2 \cdot D\left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+336\hat{q}x^2(1-x)^2 \cdot D \cdot \left(\frac{d}{2}-2\right)L^{\frac{d}{2}-3}$$
$$+120x(1-x)R \cdot L^{\frac{d}{2}-2} \tag{6.11}$$

We now write an explicit form of contributions arising from first two terms in (6.5):

$$
\begin{aligned}
&-\Sigma_\theta|_{i\hat{q}=-m} - \frac{\partial \Sigma_\theta}{\partial i\hat{q}}|_{i\hat{q}=-m} \\
&= \frac{-e^2\pi^2}{(2\pi)^4}\tilde{\theta}^2 \int_0^1 dx \left\{[2i\hat{q}(1-x)+4m][q^4x(1-x)+q^2m^2x]\ln\frac{1}{L}\right. \\
&\quad + 2m^5x^2(1-x)\ln\frac{1}{m^2x^2} + 2m^4x^2(1-x)(i\hat{q}+m)\ln\frac{1}{m^2x^2} \\
&\quad - 4m^4x(1+x)(2x-1)(i\hat{q}+m)\ln\frac{1}{m^2x^2} - (i\hat{q}+m)4m^4x(1-x^2) \\
&\quad + M[(2i\hat{q}(1-x)+4m)(q^4x(1-x)+q^2m^2x) + 2m^5x^2(1+x) \\
&\quad \left. + 2m^4x^2(1-x)(i\hat{q}+m) - 4m^4x(1+x)(2x-1)(i\hat{q}+m)]\right\} \qquad (6.12)
\end{aligned}
$$

where

$$
\tilde{\theta}^2 = \theta^2(1-d)/2(2-d) \quad \text{and} \quad M = -\ln m^2x^2 + \frac{1}{2-d/2} - \gamma \qquad (6.13)
$$

Our main confirmation is that coefficients (6.8)–(6.12) at the singular value of M are exactly cancelled. These coefficients are defined as

$$
K = K_1 + K_2 + K_3 + K_4 + K_5 \qquad (6.14)
$$

where K_1, K_2, K_3, K_4, and K_5 are arisen from (6.12), (6.8), (6.9), (6.10), and (6.11), with terms like $(m^2x^2)^{\frac{d}{2}-2}$, respectively. Thus

$$
\begin{aligned}
K_1 &= (2i\hat{q}(1-x)+4m)(q^4x(1-x)+q^2m^2x) \\
&\quad + 2m^5x^2(1+x) + 2m^4x^2(1-x)(i\hat{q}+m) \\
&\quad - 4m^4x(1+x)(2x-1)(i\hat{q}+m), \\
K_2 &= [-4m^3x(1-x)(2x-1) + 2m^3x(1+x)(6x-5)] \\
&\quad \times [2m(m+i\hat{q}) - q^2 - m^2], \\
K_3 &= [2m^2x(1-x)(x-5(x-1)) + 8m^2x(1-x^2)] \\
&\quad \times [-2m^3 - 2mq^2 + 3m^2(m+i\hat{q}) = q^2(m+i\hat{q})], \\
K_4 &= [8mx(1-x)^2 - 2mx(1-x^2)] \\
&\quad \times [(m^2+q^2)^2 - 4m^2(m^2+q^2) + (m+i\hat{q})(4m^3-4mq^2)],
\end{aligned}
$$

and

$$K_5 = [-2x(1-x)^2][(m^2+q^2)^2(m+i\hat{q}) + (m+i\hat{q})4m^2(-m^2-q^2)] \\ +(m+i\hat{q})2m(4m^3-4mq^2) + (m^2+q^2)(4m^3-4mq^2)]$$

Now one can classify these terms as follows:

$$K = (m+i\hat{q})[q^4\cdot l_1 + m^2q^2\cdot l_2 + m^4\cdot l_3] + q^4 m l_4 + m^3q^2\cdot l_5 + m^5\cdot l_6 \tag{6.15}$$

Here

$$\begin{aligned}
l_1 &= 2x(1-x)^2 - 2x(1-x)^2 \equiv 0,\\
l_2 &= 2x(1-x) + 10(1-x)^2x - 2x^2(1-x)^2\\
&\quad - 8x(1-x^2) - 32x(1-x)^2 + 8x(1-x^2)\\
&\quad + 8x(1-x)^2 - 16x(1-x)^2 - 4x(1-x)^2 \equiv 0,\\
l_3 &= 2x^2(1-x) - 4x^2(1+x) + 4x(1-x^2) + 8x^2(1-x)\\
&\quad + 8x(1-x)^2 + 4x^2(1+x) - 20x(1-x^2) + 6x^2(1-x)\\
&\quad - 30x(1-x)^2 + 24x(1-x^2) + 24x(1-x)^2\\
&\quad - 8x(1-x^2) - 2x(1-x)^2 \equiv 0,\\
l_4 &= 4x(1-x) - 2x(1-x^2) + 8x(1-x)^2 - 2x(1-x^2) - 8x(1-x)^2 \equiv 0,\\
l_5 &= 4x - 2x(1-x) + 4x^2(1-x) - 4x(1-x)^2\\
&\quad - 2x^2(1+x) + 10x(1-x^2) - 4x^2(1-x)\\
&\quad + 20x(1-x)^2 - 16x(1-x^2) + 4x(1-x^2) - 16x(1-x)^2 \equiv 0
\end{aligned}$$

and

$$\begin{aligned}
l_6 &= 2x^2(1+x) + 4x^2(1-x) - 4x(1-x)^2\\
&\quad - 2x^2(1+x) + 10x(1-x^2) - 4x^2(1-x) + 20x(1-x^2)\\
&\quad - 16x(1-x^2) - 24x(1-x)^2 + 6x(1-x^2) + 8x(1-x)^2 \equiv 0.
\end{aligned}$$

Thus, we see that as in the case of the vacuum polarization, the poles at $d = 4$ and the term $-\gamma$ in $\Gamma(2 - d/2)$ cancel in Σ_θ exactly. Multiplier (6.13) has arisen from the product

$$\Gamma\left(2-\frac{d}{2}\right)(m^2x^2)^{\frac{d}{2}-2} \rightarrow \left(\frac{1}{2-d/2} - \gamma\right)\left[1 - \left(2-\frac{d}{2}\right)\ln m^2x^2\right]$$

$$= -\ln m^2x^2 + \frac{1}{2-d/2} - \gamma = M$$

at the limit $d \to 4$. Other terms in (6.8)–(6.11) give the product $\Gamma(2-\frac{d}{2})(2-\frac{d}{2})\cdots$ which is finite at this limit. After a straightforward but tedious calculation, Eq. (6.5) becomes

$$\begin{aligned}\Sigma_\theta(q) = \Sigma_{\rm local}(q) - \frac{3\,e^2\pi^2\theta^2}{4(2\pi)^4}\int_0^1 dx \Bigg\{ &[2i\hat{q}(1-x)+4m][q^4x(1-x)+q^2m^2x] \\ &\times \ln\frac{m^2x^2}{q^2x(1-x)+m^2x} + (m+i\hat{q})\left[\frac{10}{3}m^4(1-x)(-13+34x-7x^2)\right. \\ &+q^2m^2\left(\frac{4}{3}(1-x)(95-288x+109x^2)-\frac{2}{3}(1-x)^2(16x-11)\right) \\ &\left.+\frac{2}{15}q^4(1-x)^2(-98+293x)\right] \\ &+\left(m+\frac{q^2}{m}\right)\left[4m^3(1-x)\left(\frac{569}{60}+\frac{229}{12}x+\frac{84}{15}x^2\right)\right. \\ &\left.\left.+\,2mq^2(1-x)\left(-\frac{829}{30}+\frac{143}{6}x+\frac{772}{15}x^2\right)\right]\right\} \end{aligned} \quad (6.16)$$

where $\Sigma_{\rm local}(q)$ is given by Eq. (6.7).

It should be noted that in expressions (6.8)–(6.11) there appear terms of the type:

$$108\cdot\hat{q}q^2x^3(1-x)^3\cdot L\cdot D\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-3\right)\left(\frac{d}{2}-4\right)L^{\frac{d}{2}-5}$$

that give divergence at the point $x=0$. These singularities are caused by the pole of the value $L=q^2x(1-x)+m^2x$ at the point $q^2=-m^2$,

$$L(q^2)|_{q^2=-m^2} = m^2x^2 \quad (6.17)$$

Since, contributions $\Sigma^1_\theta(q)$ due to noncommutativity of spacetime are free from infrared divergences in the presence of the factor $((p\cdot q)^2 - p^2q^2)$ in (6.3), and therefore we have omitted these fictitious divergences connected with a concrete form of the Taylor series (6.5). Indeed, in some textbooks, for example, by Bogolubov and Shirkov (1980) for removal of regularization (in our case, d-dimensional one), it is used as an another form of the Taylor series

$$\Sigma(q) - \Sigma(0) - \frac{\partial\Sigma}{\partial q^n}|_{q=0}\cdot q^n - \cdots \quad (6.18)$$

instead of (6.5). Of course, in this case, Eq. (6.17) reads

$$L(q^2)|_{q^2=0} = m^2 x, \tag{6.19}$$

and therefore, all expressions (6.8)–(6.11) are finite at the point $x = 0$. However, for this case we would obtain another expression $\Sigma'_\theta(q)$, (instead of (6.16)):

$$\Sigma'_\theta(q) = \Sigma_\theta(q) + c_1(m + i\hat{q}) + c_2 m$$

where c_1 and c_2 are some constants. In x-space last term acquires the form

$$[c_1(m - \hat{\partial}) + c_2 m]\partial^4(x),$$

disappearing at $x \neq 0$, i.e., as should be expected from the general consideration, in this term arbitrariness of the T-product (at the same time, the $\star$-product) takes place only at infinity small neighborhood of the point $x = 0$.

Finally, we write explicit form of last two terms in (6.16) for the cases of (6.18) and (6.19):

$$\begin{aligned}&(m + i\hat{q})[m^4(-4x^2(1 - x^2) + 4i_1 + 6i_2 + 8i_3 + 5i_4)\\ &\quad + 2q^2m^2(-i_2 - 4i_3 - 5i_4) + q^4 \cdot i_4]\\ &\quad + m \cdot 2(q^2 + m^2)[m^2(-i_1 - 2i_2 - 3i_3 - 2i_4) + q^2(i_3 + 2i_4)]\end{aligned} \tag{6.20}$$

where

$$i_1 = x(1 - x)(5 + 7x - 2x^2), \quad i_2 = \frac{1}{3}x(1 - x)^2(-28 - 35x - 4x^2),$$

$$i_3 = x(1 - x^2) + x(1 - x)^2\left(-\frac{41}{4} - \frac{23}{2}x + \frac{187}{12}x^2 - \frac{20}{3}x^3\right),$$

and

$$i_4 = \frac{x(1 - x)^2}{15}(8 + 574x - 646x^2 + 340x^3 - 48x^4)$$

One can see that Eq. (6.16) with (6.20) is finite at the point $x = 0$.

7. ANOMALOUS MAGNETIC MOMENTS AND CHARGE RADII IN NQED

Let us consider contributions due to noncommutativity of spacetime to the magnetic moment and the charge radius of an electron or muon in lowest order radiative corrections. Here we need to calculate the matrix element corresponding to the one-loop graph in Fig. 5.

By construction the $\star$- product of this diagram

$$e^{-ipx} \star e^{-i(p-k)(x-z)} \star e^{-(p'-p)z} \star e^{-i(p'-k)(z-y)} \star e^{-ik(y-x)} \star e^{ip'y}$$

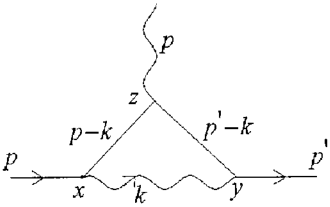


Fig. 5. One-loop diagram for the photon–lepton vertex function Γ^{μ} in NQED.

gives rise to a factor in the momentum space:

$$U = \exp\left[\frac{3}{2}\theta\sigma_{\mu\nu}k^{\mu}p^{\nu} - \theta\sigma_{\alpha\beta}k^{\alpha}p'^{\beta} + \theta\sigma_{\rho\sigma}p^{\rho}p'^{\sigma}\right] = 1 + U_1 + U_2 + U_3 \quad (7.1)$$

where

$$U_1 = \frac{3}{2}\theta\sigma_{\mu\nu}k^{\mu}p^{\nu} - \theta\sigma_{\alpha\beta}k^{\alpha}p'^{\beta} + \theta\sigma_{\rho\sigma}p^{\rho}p'^{\sigma} \quad (7.2)$$

$$U_2 = -\frac{\theta^2}{2}\left[-\frac{9}{4}\sigma_{\mu\nu}\sigma_{\chi\lambda}k^{\mu}p^{\nu}k^{\chi}p^{\lambda} - \sigma_{\alpha\beta}\sigma_{\tau\gamma}k^{\alpha}k^{\tau}p'^{\beta}p'^{\gamma}\right.$$
$$-\sigma_{\rho\sigma}\sigma_{\theta\Delta}p'^{\sigma}p^{\theta}p'^{\Delta} + 3\sigma_{\mu\nu}\sigma_{\alpha\beta}k^{\mu}k^{\alpha}p^{\nu}p'^{\beta}$$
$$-3\sigma_{\mu\nu}\sigma_{\rho\sigma}k^{\mu}p^{\nu}p'^{\sigma} + 2\sigma_{\alpha\beta}\sigma_{\rho\sigma}k^{\alpha}p'^{\beta}p^{\rho}p'^{\sigma} \quad (7.3)$$

and so on. Here $\sigma_{\mu\nu} = \gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}$.

It is easy to see that the factor (7.1) in the limit $q^2 = (p - p')^2 \to 0$ turns to the form factor (6.2), that is corresponding or coordinating prescription of the noncommutative theory. Making use of (2.6), one finds the trace of (7.1) with (7.3) in the form:

$$F_1 = \frac{1}{N(d)}TrU = 1 - \frac{\theta^2}{2}[-9((k\cdot p)^2 - k^2p^2) - 4((p'\cdot k)^2 - k^2p'^2$$
$$-4((p\cdot p')^2 - p^2p^{r2}) + 12((k\cdot p')(k\cdot p) - (p\cdot p')k^2) - 12((k\cdot p')p^2$$
$$-(k\cdot p)(p\cdot p')) + 8((k\cdot p')(p\cdot p') - (k\cdot p)p'^2)] \quad (7.4)$$

For some cases, we need the trace of terms in (7.1) which are proportional to θ^4:

$$F_2 = F_1 + \frac{\theta^4}{4}F_3 \quad (7.5)$$

where

$$F_3 = \frac{2}{-\theta^2}\mathrm{Tr}U_2^2$$
$$= \frac{81}{16}[(k\cdot p)^2 - k^2p^2]^2 + [k\cdot p')^2 - k^2p'^2]^2 + [(p\cdot p')^2 - p^2p'^2]^2$$

$$
\begin{aligned}
&-4\{-(k\cdot p)[p'^2((p\cdot p')^2-p^2p'^2)]+(k\cdot p')[(p\cdot p')((p\cdot p')^2-p^2p'^2)]\}\\
&+2\{(k\cdot p')^2[(p\cdot p')^2-p^2p'^2]-k^2[p'^2((p\cdot p')^2-p^2p^{r2})]\}\\
&-18\{(k\cdot p)k^2[2(p\cdot p')^2-p^2p'^2]+(k\cdot p')k^2[-p^2(p\cdot p')]\\
&+2p^2(k\cdot p)(k\cdot p')^2-3(p\cdot p')(k\cdot p')(k\cdot p)^2+p'^2(k\cdot p)^3\}\\
&+6\{-(k\cdot p')K^2p^2P'^2+(k\cdot p)k^2(p\cdot p')p'^2-(k\cdot p)(k\cdot p')^2\cdot(p\cdot p')\\
&+p^2(k\cdot p')^3\}+\frac{27}{2}\{-p^4(k\cdot p')k^2+k^2p^2(k\cdot p)(k\cdot p')\\
&+p^2(k\cdot p')\cdot(k\cdot p')^2-(p\cdot p')(kp)^3\}-9\{p^2p'^2(k\cdot p)k^2\\
&-p^2(p\cdot p')(k\cdot p')k^2-p'^2(kp)^3+(k\cdot p')(p\cdot p')(kp)^2\}\\
&-4\{p'^4(k\cdot p)k^2-p'^2(p\cdot p')(k\cdot p')k^2+(p\cdot p')(k\cdot p')^3\\
&-p'^2(k\cdot p)(k\cdot p')^2\}+12\{p'^2(p\cdot p')(k\cdot p)k^2-p^2p'^2(k\cdot p')k^2\\
&-(p\cdot p')(k\cdot p)(k\cdot p')^2+p^2(k\cdot p')^3\}-12\{2(p\cdot p')p'^2(kp)^2\\
&+(k\cdot p')^2(2(p\cdot p')p^2)-(p^2p'^2+3(p\cdot p')^2)(kp)(k\cdot p')\\
&+k^2[(p\cdot p')^3-(p\cdot p')p^2p'^2]\}\\
&9\{k^2p^2((p\cdot p')^2-p^2p'^2)+(k\cdot p)^2(p^2p'^2-(p\cdot p')^2)+2p^4(kp')^2\\
&-2p^2(p\cdot p')(k\cdot p)(k\cdot p')\}\\
&+4\{k^2p'^2((p\cdot p')^2-p^2p'^2)+(k\cdot p')^2((p\cdot p')^2+p^2p'^2)+2p'^4(kp)^2\\
&-4p'^2(p\cdot p')(k\cdot p)(k\cdot p')\}\\
&+9\{k^4(2(p\cdot p')^2-p^2p'^2)-4(p\cdot p')(k\cdot p)(k\cdot p')k^2+p'^2k^2(kp)^2\\
&+k^2p^2(k\cdot p')^2+(k\cdot p)^2(k\cdot p')^2\}\\
&+\frac{9}{2}\{p'p'^2\cdot k^4-p^2k^2(k\cdot p')^2-p'^2k^2(k\cdot p)^2+(k\cdot p)^2(k\cdot p')^2\}\\
&+\frac{9}{2}\{p^2((p\cdot p')^2-p^2p'^2)\cdot k^2\}-\frac{27}{2}\{p^2k^2(p\cdot p')-p^2k^2(k\cdot p)(k\cdot p')\\
&-k^2(k\cdot p)^2(p\cdot p')+(k\cdot p')(k\cdot p)^3\}\\
&-6\{[(k\cdot p)(k\cdot p')-k^2(p\cdot p')]((p\cdot p')^2-p^2p'^2)\}\\
&\quad+6\{[(k\cdot p')p^2-(p\cdot p')(k\cdot p)][(p\cdot p')^2-p^2p'^2]\}
\end{aligned}\tag{7.6}
$$

Here, we have used expression of the type of (2.8) for the calculation of the trace of multi-γ_μ-matrices.

In NQED, one-loop graph (Fig. 5) gives the matrix element:

$$\Gamma^{\mu}_{\theta}(p', p) = \int d^4k[e\gamma^{\rho}(2\pi)^4]\left[\frac{-i}{(2\pi)^4}\frac{-i(\hat{p}'-\hat{k})+m}{(2\pi)^4(p'-k)^2+m^2-i\varepsilon}\right]$$
$$\times[\gamma^{\mu}]\left[\frac{-i}{(2\pi)^4}\frac{-i(\hat{p}-\hat{k})+m}{(p-k)^2+m^2-i\varepsilon}\right][e\gamma_{\rho}(2\pi)^4]$$
$$\times\left[\frac{-i}{(2\pi)^2}\frac{1}{k^2-i\varepsilon}\right][1+F_1(\theta, p', p)] \tag{7.7}$$

where p' and p are the final and initial lepton four momenta, respectively. This integral has ultraviolet divergence, here we do use the dimensional regularization procedure.

To combine denominators, we use the Feynman parameterization prescription

$$\frac{1}{(p'-k)^2+m^2-i\varepsilon}\frac{1}{(p-k)^2+m^2-i\varepsilon}\frac{1}{k^2-i\varepsilon}$$
$$= 2\int_0^1 dx\int_0^x dy[((p'-k)^2+m^2-i\varepsilon)y+((p-k)^2+m^2-i\varepsilon)(x-y)$$
$$+(k^2-i\varepsilon)(1-x)]^{-3}$$
$$= 2\int_0^1 dx\int_0^x dy[(k-p'y-p(x-y))^2+m^2x^2+q^2y(x-y)-i\varepsilon]^{-3}.$$

Here $q = p - p'$ is the momentum transferred to the photon.

Shifting the variable of integration

$$k \to k + p'y + p(x-y) \tag{7.8}$$

the integral (7.7) becomes

$$\Gamma^{\mu}_{0}(p', p) = \frac{2i\,e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int d^4k[k^2+m^2x^2+q^2y(x-y)-i\varepsilon]^{-3}$$
$$\times\gamma^{\rho}[-i(\hat{p}'(1-y)-\hat{k}-\hat{p}(x-y))+m]\gamma^{\mu} \tag{7.9}$$
$$[-i(\hat{p}'(1-x+y)-\hat{k}-\hat{p}y)+m]\gamma_{\rho}[1+Q(k, p, p', x, y)]$$

where variable Q has arisen from Eq. (7.4) by using the shift (7.8):

$$Q = -\frac{\theta^2}{2}\{-9[(k\cdot p)^2-k^2p^2]-4[(k\cdot p')^2-k^2p'^2]+12[(k\cdot p')(k\cdot p)$$
$$-k^2(p\cdot p')]+2(k\cdot p)[2p'^2-3(p\cdot p')](2x+y-2)+2(k\cdot p')$$
$$\times[3p^2-2(p\cdot p')](2x+y-2)+(p^2p'^2-(p\cdot p')^2)(y-2+2x)^2\} \tag{7.10}$$

Next step is a Wick rotation, replace the volume element $d^4k_E = \Omega_d k^{d-1}dk$ and use the formulas (5.6)–(5.12) and (5.13). Putting this all together, Eq. (7.9) now becomes

$$\Gamma^{\mu}_{\theta}(p', p) = \frac{-2\,e^2\Omega_d}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^{\infty} k^{d-1}\,dk[k^2+L]^{-3}$$
$$\times\Big\{[-k^2\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma_{\sigma}\gamma_{\rho}/d] + \gamma^{\rho}[-i\left(\hat{p}'(1-y) - \hat{k} - \hat{p}(x-y)\right)$$
$$+ m]\gamma^{\mu}[-i(\hat{p}'(1-x+y) - \hat{k} - \hat{p}y) + m]\gamma_{\rho}$$
$$- \frac{\theta^2}{2}[A+B+C+D][Q_1+Q_2+Q_3]\Big\} \tag{7.11}$$

where we have used short notation

$$L = m^2x^2 + q^2y(x-y),$$
$$A = \gamma^{\rho}[-i(\hat{p}'(1-y) - \hat{p}(x-y)) + m]\gamma^{\mu}[-i(\hat{p}'(1-x+y) - \hat{p}y) + m]\gamma_{\rho},$$
$$B = -\gamma^{\rho}\hat{k}\gamma^{\mu}\hat{k}\gamma_{\rho},$$
$$C = \gamma^{\rho}[-i(\hat{p}'(1-y) - \hat{p}(x-y)) + m]\gamma^{\mu}(i\hat{k})\gamma_{\rho}$$
$$D = \gamma^{\rho}(i\hat{k})\gamma^{\mu}[-i(\hat{p}'(1-x+y) - \hat{p}y) + m]\gamma_{\rho}, \tag{7.12}$$

and

$$Q_1 = -9[(k\cdot p)^2 - k^2p^2] - 4[(k\cdot p')^2 - k^2p'^2]$$
$$+ 12[(k\cdot p')(k\cdot p) - k^2(p\cdot p')]$$
$$Q_2 = 2(k\cdot p)[2p'^2 - 3(p\cdot p')](2x+y-2) + 2(k\cdot p')[3p^2 - 2(p\cdot p')]$$
$$\times(2x+y-2)$$
$$Q_3 = (p^2p'^2 - (p\cdot p')^2)(y-2+2x)^2 \tag{7.13}$$

To carry out integration over the variable k, we need to calculate the following type of expressions:

$$-\gamma^{\rho}\hat{k}\gamma^{\mu}\hat{k}\gamma_{\rho}(k\cdot p)^2 = -\frac{k^4}{d(d+2)}[p^2((2-d)^2 - 2(2-d))\gamma^{\mu}$$
$$+4(2-d)p^{\mu}\hat{p}]|_{d\to 4} = -\frac{1}{3}k^4(p^2\gamma^{\mu} - p^{\mu}\hat{p}),$$
$$-\gamma^{\rho}\hat{k}\gamma^{\mu}\hat{k}\gamma_{\rho}\cdot k^2 = -\frac{(2-d)^2}{d}k^4\gamma^{\mu}|_{d\to 4} = -k^4\gamma^{\mu}$$

and

$$-\gamma^\rho \hat{k}\gamma^\mu \hat{k}\gamma_\rho (k\cdot p)(k\cdot p') = -k^4\left[(p\cdot p')\gamma^\rho\gamma^\beta\gamma^\mu\gamma_\beta\gamma_\rho \right. \\ \left. +\gamma^\rho \hat{p}\gamma^\mu \hat{p}'\gamma_\rho + \gamma^\rho \hat{p}'\gamma_\rho\right]/d(d+2) \quad (7.14)$$

and so on. Further, using the γ^μ-algebra, one can transform last two terms in (7.14) in the form:

$$\Delta = \gamma^\rho \hat{p}\gamma^\mu \hat{p}'\gamma_\rho + \gamma^\rho \hat{p}'\gamma^\mu \hat{p}\gamma_\rho \\ = (2-d)[2p^\mu \hat{p}' + 2p'^\mu \hat{p} - 2(p\cdot p')\gamma^\mu]$$

After some calculations, we have

1) $$A\cdot Q_1 = A\left(1-\frac{1}{d}\right)k^2(3p-2p')^2,$$

2) $$[A+B]Q_3 = [A+B](p^2p'^2 - (p\cdot p')^2)(y-2+2x)^2$$

is almost local theory with $\theta^2/2$.

3) $$BQ_1 = -k^4\left\{-\frac{(2-d)^2 - 2(2-d)}{d(d+2)}(3p-2p')^2\gamma^\mu \right. \\ + \frac{(2-d)^2}{d}(3p-2p')^2\gamma^\mu + \frac{4(2-d)}{d(d+2)}[-9p^\mu\hat{p} \\ \left. -4p'^\mu\hat{p}' + 6\hat{p}^\mu\hat{p}' + 6p'^\mu\hat{p}]\right\} \quad (7.15)$$

and

$$(C+D)Q_2 \\ = \frac{ik^2}{d}\cdot 2(2x+y-2)\left\{\gamma^\rho[-i(\hat{p}'(1-y) - \hat{p}(x-y)) + m]\gamma^\mu\hat{p}\gamma_\rho \right. \\ \left. +\gamma^\rho\hat{p}\gamma^\mu[-i(\hat{p}(1-x+y) - \hat{p}'y) + m]\gamma_\rho\right\}\cdot(2p'^2 - 3(p\cdot p')) \\ + \frac{ik^2}{d}\cdot 2(2x+y-2)\left\{\gamma^\rho[-i(\hat{p}'(1-y) - \hat{p}(x-y)) + m]\gamma^\mu\hat{p}'\gamma_\rho \right. \\ \left. +\gamma^\rho\hat{p}'\gamma^\mu[-i(\hat{p}(1-x+y) - \hat{p}'y) + m]\gamma_\rho\right\}\cdot(3p'^2 - 2(p\cdot p')) \quad (7.16)$$

As in the local theory, we are interested here only in the matrix element $\bar{u}(p')\Gamma^\mu_\theta(p', p)u(p)$ of the vertex function between Dirac spinors that satisfy the relations

$$\bar{u}(p')(i\hat{p}' + m) = 0, \qquad [i\hat{p} + m]u(p) = 0$$

We are able therefore to simplify this expression by using the anticommutation relations of the Dirac matrices to move all factors p' to the left and all factors p to the right, replacing them when they arrive on the left or right with im. We take into account the following standard relations between two Dirac spinors $\bar{u}(p')$ and $u(p)$:

$$\begin{aligned}
a_1 &= -(1-y)(1-x+y)\gamma^\rho \hat{p}'\gamma^\mu \hat{p}\gamma_\rho \\
&= -2(1-y)(1-x+y)[-3m^2\gamma^\mu - q^2\gamma^\mu - 2im(p'^\mu + p^\mu)], \\
a_2 &= (x-y)(1-x+y)\gamma^\rho \hat{p}\gamma^\mu \hat{p}\gamma_\rho = (x-y)(1-x+y)[-4imp^\mu - 2m^2\gamma^\mu], \\
a_3 &= -im(1-x+y)\gamma^\rho\gamma^\mu \hat{p}\gamma_\rho = -4im(1-x+y)p^\mu, \\
b_1 &= y(1-y)\gamma^\rho \hat{p}'\gamma^\mu \hat{p}'\gamma_\rho = y(1-y)[-4imp'^\mu - 2m^2\gamma^\mu], \\
b_2 &= -y(x-y)\gamma^\rho \hat{p}\gamma^\mu \hat{p}'\gamma_\rho = -2m^2 y(x-y)\gamma^\mu \\
b_3 &= im\gamma^\rho\gamma^\mu \hat{p}\gamma_\rho \cdot y = 4imyp^\mu \\
c_1 &= -im(1-y)\gamma^\rho \hat{p}'\gamma^\mu\gamma_\rho = -4im(1-y)p'^\mu \\
c_2 &= im(x-y)\gamma^\rho \hat{p}\gamma^\mu\gamma_\rho = 4im(x-y)p^\mu \\
c_3 &= m^2\gamma^\rho\gamma^\mu\gamma_\rho = -2m^2\gamma^\mu
\end{aligned}$$

We sum up these expressions and obtain

$$\begin{aligned}
A = {} & 2m^2\gamma^\mu(x^2-4x+2) + 2(1-y)(1-x+y)q^2\gamma^\mu \\
& + 4im(y-x+xy)p'^\mu + 4im(x^2-xy-y)p'
\end{aligned} \tag{7.17}$$

This is the result of the local theory case. To simplify expression due to noncommutativity of spacetime, we have

$$\begin{aligned}
d_1 &= -i(1-y)\gamma^\rho \hat{p}'\gamma^\mu \hat{p}\gamma_\rho = -2i(1-y)[-3m^2\gamma^\mu - q^2\gamma^\mu - 2im(p'^\mu + p^\mu)], \\
d_2 &= i(x-y)\gamma^\rho \hat{p}\gamma_\rho = i(x-y)[-4imp^\mu - 2m^2\gamma^\mu], \\
d_3 &= m\gamma^\rho\gamma^\mu \hat{p}\gamma_\rho = 4mp^\mu, \\
e_1 &= -i\gamma^\rho \hat{p}\gamma^\mu \hat{p}\gamma_\rho \cdot (1-x+y) = -(1-x+y)[-4imp^\mu - 2m^2\gamma^\mu], \\
e_2 &= iy\gamma^\rho \hat{p}\gamma^\mu \hat{p}' = 2iym^2\gamma^\mu, \\
e_3 &= m\gamma^\rho \hat{p}\gamma^\mu\gamma_\rho = 4mp^\mu, \\
D_1 &= -i(1-y)\gamma^\rho \hat{p}'\gamma^\mu \hat{p}'\gamma_\rho = -i(1-y)[-4imp'^\mu - 2m^2\gamma^\mu], \\
D_2 &= i(x-y)\gamma^\rho \hat{p}\gamma^\mu \hat{p}'\gamma_\rho = i2\cdot(x-y)m^2\gamma^\mu, \\
D_3 &= m\gamma^\rho\gamma^\mu \hat{p}'\gamma_\rho = 4mp'^\mu, \\
E_1 &= -i(1-x+y)\gamma^\rho \hat{p}'\gamma^\mu \hat{p}\gamma_\rho = -2i(1-x+y)[-3m^2\gamma^\mu
\end{aligned}$$

$$-q^2\gamma^\mu - 2im(\hat{p}'^\mu + p^\mu)],$$

$$E_2 = iy\gamma^\rho \hat{p}'\gamma^\mu \hat{p}'\gamma_\rho = iy[-4im\hat{p}'^\mu - 2m^2\gamma^\mu],$$

and

$$E_3 = m\gamma^\rho \hat{p}'\gamma^\mu\gamma_\rho = 4m\hat{p}'^\mu$$

Last terms from d_1 to E_3 are arisen from the expression which is proportional to θ_2.

In the noncommutative quantum electrodynamics, the vertex function corresponding to the diagram shown in Fig. 5 takes the form by means of short notation:

$$\Gamma^\mu_\theta(p', p) = \frac{-2e^2\Omega_d}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty k^{d-1}\,dk \times [A + B + C + D][k^2 + L]^{-3}\left[1 - \frac{\theta^2}{2}(Q_1 + Q_2 + Q_3)\right] \tag{7.18}$$

Here, L, A, B, C, D, Q_1, Q_2, and Q_3 are given by expressions (7.12) and (7.13). According to above calculations,

$$\Lambda = \bar{u}(p')Au(p) = 2m^2(x^2 - 4x + 2)\gamma^\mu$$
$$2(1-y)(1-x+y)q^2\gamma^\mu - 2imx(1-x)[p^\mu + \hat{p}'^\mu],$$
$$\bar{u}(p')Bu(p) = -\frac{(2-d)^2}{d}k^2\cdot\gamma^\mu,$$
$$\bar{u}(p')AQ_1u(p) = k^2\left(1 - \frac{1}{d}\right)(3q^2 - 7m^2)\Lambda, \tag{7.19}$$

$$\bar{u}(p')(A+B)Q_3u(p) = \left[-\frac{(2-d)^2}{d}k^2\cdot\gamma^\mu + \Lambda\right](y - 2 + 2x)^2 \times\left(-m^2q^2 - \frac{q^4}{4}\right)|_{d\to4}$$
$$= (-k^2\gamma^\mu + \Lambda)(y - 2 + 2x)^2\left(-m^2q^2 - \frac{q^4}{4}\right),$$

$$\bar{u}(p')BQ_1u(p) = -k^4\left[\frac{2-d}{d}\frac{4+d-d^2}{d+2}(3p - 2p')\gamma^\mu - i\frac{m}{2}\frac{4(2-d)}{d(d+2)}(p^\mu + p'^\mu)\right]|_{d\to4}$$
$$= -k^4\left[\frac{2}{3}(3q^2 - 7m^2)\gamma^\mu + \frac{im}{6}(p^\mu + p'^\mu)\right] \tag{7.20}$$

$$\bar{u}(p')(C+D)Q_3u(p) = 10i\frac{m}{d}k^2q^2(p^\mu + p'^\mu)(2x+y-2)(2x-1)$$
$$-\frac{2}{d}k^2\cdot\gamma^\mu(2x+y-2)[4m^2q^2(5-2x-y)+q^4(5-y-2x)] \quad (7.21)$$

Notice that in Eqs. (7.19)–(7.21) we have exploited the symmetry of the vertex function (or the diagram) under the reflection $p \to p'$ (or $y \to x-y$) that gives the factor $p^\mu + p'^\mu$ exactly.

We next use the integral formula (5.16), the Gamma-function algebra (5.6)–(5.12), and the limiting procedure like (5.14) and (5.30) for removal of the d-dimensional regularization as before.

According to the local theory there are other diagrams that need to be taken into account. There is the zeroth-order term γ^μ in Γ^μ_θ. The term proportional to Z_2-1 in the contribution term (4.7) gives a term in Γ^μ_θ:

$$\Gamma^\mu_{\theta\mathcal{L}2} = (Z_2-1)\gamma^\mu \quad (7.22)$$

Also, the effect of insertions of corrections to the external photon propagator is a term:

$$\Gamma^\mu_{\theta,\text{vac,pol}}(p',p) = \frac{1}{(p'-p)^{2-i\varepsilon}}\Pi^{\mu\nu}_\theta(p'-p)\gamma_\nu \quad (7.23)$$

The form of each of these terms (7.18), (7.22), and (7.23) is in agreement with the general rule:

$$\bar{u}(p')\Gamma^\mu(p',p)u(p) = \bar{u}(p')\left[\gamma^\mu F_\theta(q^2) - \frac{i}{2m}(p+p')^\mu G_\theta(q^2)\right]u(p) \quad (7.24)$$

To order e^2, the form factors are

$$F_\theta(q^2) = Z_2 + \Pi_\theta(q^2) + F_{\text{local}}(q^2) + F_{1\theta}(q^2) \quad (7.25)$$

and

$$G_\theta(q^2) = G_{\text{local}}(q^2) + G_{1\theta}(q^2) \quad (7.26)$$

where $\Pi_\theta(q^2)$ is the vacuum polarization function (5.33),

$$f_{\text{local}}(q^2) = \frac{-2\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\left[\ln\frac{m^2x^2+q^2y(x-y)}{m^2x^2}\right.$$
$$\left.+\frac{m^2(x^2-4x+2)+q^2(1-y)(1-x+y)}{m^2x^2+q^2y(x-y)}\right] \quad (7.27)$$

$$G_{\text{local}}(q^2) = \frac{-e^2m^2}{4\pi^2}\int_0^1 dx\int_0^x dy\frac{x(1-x)}{m^2x^2+q^2y(x-y)} \quad (7.28)$$

$$F_{1\theta}(q^2) = -\frac{\theta^2}{2}\left\{F_{\text{local}}\cdot(y-2+2x)^2\left(m^2q^2+\frac{q^4}{4}\right)\right.$$

$$-\frac{4\pi^2 e^2}{(2\pi)^4}\int_0^1 dv\int_0^x dy\left[-\ln\frac{m^2x^2+q^2y(x-y)}{m^2x^2}\right.$$
$$\left(\alpha_1 m^4+\alpha_2 m^2q^2+q^4\left(\frac{9}{2}(1-x)+\frac{21}{2}y(x-y)\right.\right.$$
$$\left.\left.+\frac{3}{4}(y-2+2x)(-4+y+2x)\right)\right)+m^2q^2y(x-y)\frac{\alpha_1}{x^2}$$
$$\left.\left.\times\left(1-\frac{1}{2x^2}\frac{q^2}{m^2}y(x-y)\right)+\frac{y(x-y)}{x^2}\alpha_2\cdot q^4\right]\right\} \tag{7.29}$$

and

$$G_{1\theta}(q^2)=-\frac{\theta^2}{t}\left\{G_{\rm local}\cdot(y-2+2x)^2\left(m^2q^2+\frac{1}{4}q^4\right)-\frac{16\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\right.$$
$$\times\left[-\ln\frac{m^2x^2+q^2y(x-y)}{m^2x^2}\cdot(\beta_1m^4+\beta_2m^2q^2)+m^2q^2y(x-y)\frac{\beta_1}{x^2}\right.$$
$$\left.\left.\times\left(1-\frac{1}{2x^2}\frac{q^2}{m^2}y(x-y)\right)+\frac{y(x-y)}{x^2}\beta_2\cdot q^4\right]\right\} \tag{7.30}$$

Here

$$\alpha_1=-\frac{7}{2}(7x^2-6(2x-1)),$$
$$\alpha_2=-\frac{3}{2}(1-x)(1+3x)+\frac{3}{2}x(4x-3)-\frac{49}{2}y(x-y)$$
$$+3(y-2+2x)(-4+4+2x),$$
$$\beta_1=-(21x(1-x)+x^2),$$
$$\beta_2=9x(1-x)-5(2x-1)(2x+y-2)-y(x-y)$$

We see that Eq. (7.28) is finite. It makes to calculate the anomalous magnetic moment. We know that it is only γ^μ term that contributes to the magnetic moment, so the effect of radiative and noncommutative corrections is to multiply the Dirac value $e/2m$ of the magnetic moment by a factor $F_\theta(0)$. But the definition of e as the true lepton charge requires that

$$F_\theta(0)+G_\theta(0)=1 \tag{7.31}$$

so the magnetic moment may be expressed as

$$\mu_\theta=\frac{e}{2m}(1-G_\theta(0)) \tag{7.32}$$

From Eqs. (7.28) and (7.30), we find

$$-G_\theta(0) = \frac{e}{8\pi^2} = 0.001161 \tag{7.33}$$

This is the famous $\alpha/2\pi$ correction first obtained by Schwinger (1948). From explicit forms (7.29) and (7.30) it follows that

$$F_{1\theta}(0) = G_{1\theta}(0) = 0 \tag{7.34}$$

and therefore the charge-non-renormalization condition (7.31) and the magnetic moment (7.32) do not change in the noncommutative quantum electro-dynamics, at least up to the order of θ^2. However, by using the next Taylor series (7.6), one can verify that correction due to the noncommutativity of spacetime to the anomalous magnetic moment (7.33) turns to zero at least for fourth order in θ. This assertion is valid for any order in θ, since in the limit $q^2 = (p - p')^2 \to 0$, the form factor (7.1) goes to exp $[\frac{i}{2}\theta\sigma_{\mu\nu}k^\mu p^\nu]$, and therefore its trace is $\cos[\theta\sqrt{(k \cdot p)^2 - k^2 p^2}$. In this limiting case, the shift (7.8) becomes

$$k \to k + p'y + p(x - y) \to k + px$$

which gives rise $k \cdot p \to kp - m^2 x$ and $k^2 \cdot p^2 \to -m^2 k^2 - 2(k \cdot p) \cdot m^2 x + m^4 x^2$, so that $(kp)^2 - k^2 p^2 \to (k \cdot p)^2 + m^2 k^2$. Therefore to order e^2, Eqs. (7.33) and (7.31) do not change and remain as in the local theory, i.e., $F_{1\theta}(0) = G_{1\theta}(0) = 0$ for any order in θ.

Of course, this assertion is valid only for the first radiative corrections in e^2 to the magnetic moment. Even in just the next order, fourth order in e, there are so many terms that the calculations become quite complicated. However, because of the large muon–electron mass ratio, there is one fourth-order term in the magnetic moment of the muon that is somewhat larger than any of the others. It arises from the insertion of an electron loop in the virtual photon line of the second-order diagram, as shown in Fig. 6.

The effect of this electron loop is to change the photon propagator $1/k^2$ in Eq. (7.7) to $(1 + \Pi_\theta^e(g^2))/k^2$, where $\Pi_\theta^e(q^2)$ is given by Eq. (5.33), but with the

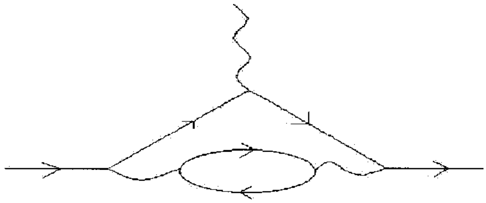


Fig. 6. A two-loop diagram for the muon-magnetic moment in NQED.

mass m taken as the electron mass:

$$\Pi^e_\theta(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\left\{\left[x(1-x) - \frac{\theta^2}{16}\left(2m_e^4 + 8m_e^2 q^2 x(1-x) - 10x^2(1-x)^2 q^4\right)\right]\right.$$
$$\left.\times \ln\left(1 + \frac{q^2 x(1-x)}{m_e^2}\right) + \frac{\theta^2}{16}\left(2m_e^2 q^2 x(1-x) - 9x^2(1-x)^2 q^4\right)\right\}$$

Inspection of Eq. (7.28) shows that in calculating the muon magnetic moment the effective cutoff on the virtual photon momentum q is m_μ. The ratio m_μ/m_e is so large that for q^2 of order m_μ^2 we may approximate

$$\Pi^e_\theta(q^2) \simeq \frac{e^2}{2\pi^2}\int_0^1 dx\left\{x(1-x)\ln(m_\mu/m_e)\right.$$
$$- \frac{\theta^2}{16}\left(2m_e^4 + 8m_e^2 m_\mu^2 x(1-x) - 10x^2(1-x)^2 m_\mu^4\right)\ln\left(m_\mu^2/m_e^2\right)$$
$$\left. - \frac{\theta^2}{16}\left(2m_e^2 m_\mu^2 x(1-x) + 9x^2(1-x)^2 m_\mu^4\right)\right\} \tag{7.35}$$

with the neglected terms having coefficients of order unity in place of $\ln(m_\mu^2/m_e^2)$. Since this is a constant, the change in $-G_\theta(0)$ produced by adding an electron loop in the virtual photon line is simply given by multiplying our previous result (7.33) for $-G_\theta(0)$ by Eq. (7.35), so that now

$$\mu_\mu = \frac{e}{2m_\mu}\left\{\frac{e^2}{8\pi^2} + \frac{e^4}{16\pi^4}\left[\frac{1}{6}\ln\left(m_\mu^2/m_e^2\right) - \frac{\theta^2}{16}\left(2m_e^4 - \frac{4}{3}m_e^2 m_\mu^2 - \frac{1}{3}m_\mu^4\right)\right.\right.$$
$$\left.\left.\times\ln\left(m_\mu^2/m_e^2\right) - \frac{\theta^2}{16}\left(\frac{1}{3}m_e^2 m_\mu^2 + \frac{3}{10}m_\mu^4\right)\right]\right\} \tag{7.36}$$

The present experimental values of the anomalous magnetic moment of muon (Carey *et al.*, 1999, Particle Data Group, 2002)

$$\mu_\mu = 1.00011659160 \pm 6.10^{-10} \tag{7.37}$$

are reliably confirmed by local quantum electrodynamics (Czarnecki and Marciano, 2001; Hughes and Kinoshita, 1999; Kinoshita, 2001). It is natural to suppose that the absolute value of the contributions calculated here should be of an order or not greater than the experimental errors. This makes it possible to establish the following restrictions on the parameter θ of the non-commutativity of spacetime:

$$\theta \lesssim 7\cdot 10^{-32} m^2 \tag{7.38}$$

Now let us consider the other form factor (7.25). To satisfy the charge-non-renormalization condition (7.31), it is necessary that Z_2 take the value

$$Z_2 = 1 + \frac{e^2}{8\pi^2} + \frac{2\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \cdot \frac{x^2 - 4x + 2}{x^2} \tag{7.39}$$

Inserting Eq. (7.39) back into Eq. (7.25) gives

$$\begin{aligned} F_\theta(q^2) = 1 + \frac{e^2}{8\pi^2} + \Pi_\theta(q^2) + \frac{2\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \\ \times \left\{ \frac{-m^2[x^2 - 4x + 2] - q^2(1-y)(1-x+y)}{m^2x^2 + q^2 y(x-y)} + \frac{x^2 - 4x + 2}{x^2} \right. \\ \left. - \ln \left[\frac{m^2x^2 + q^2y(x-y)}{m^2x^2} \right] \right\} + F_{1\theta}(q^2) \end{aligned} \tag{7.40}$$

where $F_{1\theta}(q^2)$ is given by Eq. (7.29). However, we see that the integral over x and y now diverges logarithmically at $x = 0$ and $y = 0$, because there are two powers of x and/or y in the denominators, and just two differentials $dx\, dy$ in the numerator. This divergence can be traced to the vanishing of the denominator $[k^2 + m^2x^2 + q^2y(x-y)]^{-3}$ in Eq. (7.18) at $x = 0$, $y = 0$, and $k = 0$. As in the local theory, because this infinity comes from the region of small rather than large k, it is termed an infrared divergence rather than an ultraviolet divergence. This divergence has arisen only from the local part of the noncommutative theory. Further, we shall continue our calculation by simply introducing a fictitious photon mass μ to cut off the infrared divergence in $F_\theta(q^2)$.

As mentioned above, we know from the Ward identity that $F_\theta(0) = 1 - G_\theta(0) = 1 + e^2/8\pi^2$, so let us consider the first derivative $F'_\theta(q^2)$ at $q^2 = 0$. According to Eq. (7.40) with $L_\mu = \mu^2(1-x) + L = \mu^2(1-x) + m^2x^2 + q^2y(x-y)$, $m^2x^2 \to m^2x^2 + \mu^2(1-x)$ in its denominator, this is

$$F'_\theta(0) = \Pi'_\theta(0) + F'_{\text{local}}(0) + F'_{1\theta}(0) \tag{7.41}$$

The vacuum polarization contribution is given by Eq. (5.33) as

$$\Pi'_\theta(0) = \frac{e^2}{60\pi^2 m^2} \tag{7.42}$$

Then the term $\Pi'_\theta(0) + F'_{\text{local}}(0)$ gives

$$\Pi'_\theta(0) + F'_{\text{local}}(0) = \frac{e^2}{24\pi^2 m^2} \left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{1}{4} \right] \tag{7.43}$$

with the term 2/5 the correction of vacuum polarization. While the term $F_{1\theta}(q^2)$ due to the noncommutativity of spacetime yields

$$F'_{1\theta}(0) = -\frac{1}{2}\theta^2 m^2 \left(1 + \frac{e^2}{8\pi^2}\right) \tag{7.44}$$

On the other hand, Eqs. (7.28) and (7.30) show that $G_\theta(q^2)$ has a finite derivative at $q^2 = 0$

$$G'_\theta(q^2) = G'_{\text{local}}(0) + G'_{1\theta}(0) = \frac{e^2}{48\pi^2 m^2} + G'_{1\theta}(0) \tag{7.45}$$

where

$$\begin{aligned} G'_{1\theta}(0) &= -\frac{\theta^2}{2} m^2 \left[-\frac{e^2 m^2}{4\pi^2} \int_0^1 dx \int_0^x dy \frac{x(1-x)}{m^2 x^2}(y - 2 + 2x)^2\right] \\ &= \frac{e^2}{4\pi^2}\frac{\theta^2}{2} m^2 \cdot \frac{31}{36} \end{aligned} \tag{7.46}$$

These results are most conveniently expressed in terms of the charge form factor $f_\theta(q^2)$, defined by the vertex function

$$\begin{aligned} \bar{\mu}(\mathbf{p}', \sigma')\Gamma^\mu(p', p)(\mathbf{p}, \sigma) = \bar{\mu}(\mathbf{p}', \sigma') \bigg[& \gamma^\mu f_\theta(q^2) + \frac{i}{2} i [\gamma^\mu, \gamma^\nu] \\ & \times (p' - p)_\nu f_{1\theta}(q^2) \bigg] u(\mathbf{p}, \sigma) \end{aligned} \tag{7.47}$$

where

$$f_\theta(q^2) = F_\theta(q^2) + G_\theta(q^2) \tag{7.48}$$

For $|q^2| \ll m^2$, this form factor is approximately

$$\begin{aligned} f_\theta(q^2) \simeq 1 &+ \frac{e^2}{24\pi^2}\left(\frac{q^2}{m^2}\right)\left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{3}{4}\right] \\ &- \frac{1}{2}\theta^2 m^2 q^2 \left(1 + \frac{e^2}{8\pi^2} - \frac{e^2}{4\pi^2} \cdot \frac{31}{36}\right) \end{aligned} \tag{7.49}$$

This may be expressed in terms of a charge radius a_θ defined by the limiting behavior of the charge form factor for $q^2 \to 0$:

$$f_\theta(q^2) \to 1 - \frac{1}{6} q^2 a_\theta^2 \tag{7.50}$$

Thus, the charge radius of the electron in the noncommutative quantum electrodynamics takes the form

$$a_\theta^2 = -\frac{e^2}{4\pi^2 m^2}\left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{3}{4}\right] + 3\theta^2 m^2 \left(1 - \frac{e^2}{4\pi^2} \cdot \frac{13}{36}\right) \tag{7.51}$$

We know that for electrons in atoms the role of the photon mass μ is played by an effective infrared cutoff that is much less than m, so the logarithm here is large and negative, yielding a positive value for a_θ^2. Last term in (7.51) is small contribution with respect to first one.

8. THE CAUSALITY CONDITION AND UNITARITY OF THE $S_\star$-MATRIX IN NONCOMMUTATIVE QUANTUM FIELD THEORY (NQFD)

The principles of causality and unitarity of the S-matrix in QFT are the basis of all approaches in the elementary particle theory which make claims to self-consistency and physical acceptability. Therefore, the proof of the unitarity and causality condition is crucial in constructing various models of QFT. These problems were considered in detail by Bogolubov and Shirkov (1980) and Efimov (1977), (see also Namsrai, 1986) in both the local and nonlocal cases, respectively. In this section we study spacetime properties of some functions, and the causality condition and unitarity of the $S_\star$-matrix in NQFT.

8.1. Space-Time Properties of Some Functions in the Noncommutative Space-Time

To study spacetime properties of the $S_\star$-matrix in NQFT, it is necessary to consider the local properties of test functions and generalized functions in noncommutative spacetime.

Definition 8.1. Any smooth and generalized functions $f(x)$ can be defined in noncommutative spacetime by means of the $\star$-product:

$$f_\theta(x) = f^{1/2}(x) \star f^{1/2}(x) = e^{\frac{1}{2}\ln f(x)} \star e^{\frac{1}{2}\ln f(x)} \tag{8.1}$$

and its the covariant $\star$-product reads

$$f_\theta(x) = \cos\mathrm{h}\theta\sqrt{\left(\partial_\rho^x \cdot \partial_y^\rho\right)^2 - \Box_x\Box_y}\, f^{1/2}(x)\, f^{1/2}(y)|_{y=x} \tag{8.2}$$

Differential and integral calculuses can be also formulated as the usual case (for detail, see Section 9):

$$\int d^4x \star f(x) = \int d^4x \left[\Lambda_{xy}\, f^{1/2}(x)\, f^{1/2}(y)|_{y=x}\right] \tag{8.3}$$

and

$$\frac{\partial}{\partial x^\nu} \star f(x) = \frac{\partial}{\partial x^\nu}\left[\Lambda_{xy}\, f^{1/2}(x)\, f^{1/2}(y)|_{y=x}\right] \tag{8.4}$$

where we have used the short notation

$$\Lambda_{xy} = \cos\mathrm{h}\,\theta\sqrt{\left(\partial_\rho^x\cdot\partial_y^\rho\right)^2 - \Box_x\Box_y} \tag{8.5}$$

In this case, the variational differential defines as

$$\frac{\delta}{\delta f(y)} f_\star^2(x) = \frac{\delta}{\delta f(y)}\left[f(x)(\star)_c f(x)\right] = 2\Lambda_{xy}\delta^4(x-y) f(y)|_{y=x}$$

or

$$\frac{\delta}{\delta f(y)}(\star)_c f(x) = \Lambda_{xy}\delta^4(x-y) \tag{8.6}$$

However

$$\Lambda_{xy}\delta^4(x-y) = \delta^4(x-y)$$

and therefore

$$\frac{\delta}{\delta f(y)}(\star)_c f(x) = \frac{\delta}{\delta f(y)} f(x) = \delta^4(x-y) \tag{8.7}$$

Moreover, there exist obvious equalities:

$$\int d^4y\; f(y)(\star)_c\delta^4(x-y) = f(x)$$

$$\int d^4z\;\delta^4(x-z)(\star)_c\delta^4(z-y) = \delta^4(x-y) \tag{8.8}$$

As an example, consider a function of the finite support in spacetime, say the well-known discontinuous function $\theta(l^2 - x_E^2)$, where $x_E^2 = x_0^2 + \mathrm{x}^2$. This function is located inside the four spheres defined by $x_E^2 = l^2$ or in the hyperboloid: $x^2 = l^2$, $x^2 = -x_0^2 + \mathrm{x}^2$. By the definition of the $\star$-product, in noncommutative spacetime form of this function is changed

$$\theta(l^2 - x^2) \to \theta_\theta(l^2 - x^2) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau\,\frac{e^{i\tau l^2}}{\tau - i\varepsilon} e^{-\frac{i\tau x^2}{2}} \star e^{-\frac{i\tau x^2}{2}} \tag{8.9}$$

Here we will use the covariant $\star$-product (2.12). Then, result reads

$$I_\theta(x^2) = e^{-\frac{i\tau x^2}{2}}(\star)_c\, e^{-\frac{i\tau x^2}{2}} = \Lambda_{xy}\, e^{-\frac{x^2}{2\Delta}}\, e^{-\frac{y^2}{2\Delta}}|_{y=x}$$
$$(\Delta = 1/i\tau) \tag{8.10}$$

After some direct calculations, one gets

$$\Box_x\Box_y\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}} = \lambda_1(x,y)\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}} \tag{8.11}$$

$$\left(\partial_x^\mu\,\partial_\mu^y\right) e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}} = \lambda_2(x,y)\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}} \tag{8.12}$$

where

$$\lambda_1(x,y) = \frac{16}{\Delta^4}x^2y^2 - \frac{32}{\Delta^3}(x^2+y^2) + \frac{64}{\Delta^2} \tag{8.13}$$

and

$$\lambda_2(x,y) = \frac{16}{\Delta^4}(x\cdot y)^2 - \frac{8}{\Delta^3}(x^2+y^2) + \frac{16}{\Delta^2} \tag{8.14}$$

By using these equalities, one can calculate higher order of differentials in (8.10). For example,

$$\Box_x\Box_y\Box_x\Box_y\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}}|_{y=x} = \lambda_3(x,y)\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}}|_{y=x} \tag{8.15}$$

$$\begin{aligned}\left(\partial_x^\rho\partial_\rho^y\right)^2\,\Box_x\Box_y\, e^{-\frac{x^2}{\Delta}-\frac{y^2}{\Delta}}|_{y=x} &= \Box_x\Box_y\left(\partial_x^\rho\partial_\rho^y\right)^2\, e^{-\frac{x^2}{\Delta}-\frac{y^2}{\Delta}}|_{y=x}\\ &= \lambda_4(x,y)\, e^{-\frac{x^2}{\Delta}}\, e^{-\frac{y^2}{\Delta}}|_{y=x}\end{aligned} \tag{8.16}$$

$$\left(\partial_x^\rho\partial_\rho^y\right)^2\left(\partial_x^\chi\partial_\chi^y\right)^2\, e^{-\frac{x^2}{\Delta}-\frac{y^2}{\Delta}}|_{y=x} = \lambda_5(x,y)\, e^{-\frac{x^2}{\Delta}-\frac{y^2}{\Delta}}|_{y=x}$$

Here coefficients $\lambda_3(x,y)$ and $\lambda_4(x,y)$ are given by

$$\begin{aligned}\lambda_3(x,y)|_{y=x} &= \lambda_3(x,x) = 6\cdot128\cdot12\cdot\Delta^{-4} - 128\cdot36\cdot4\cdot\Delta^{-5}\cdot x^2\\ &\quad + 128\cdot12\cdot8\cdot\Delta^{-6}(x^2)^2 - 6\cdot512\Delta^{-7}(x^2)^3 + 256\Delta^{-8}(x^2)^4,\end{aligned} \tag{8.17}$$

$$\begin{aligned}\lambda_4(x,y)|_{y=x} &= \lambda_4(x,x) = 2304\cdot\Delta^{-4} - 4608\Delta^{-5}x^2\\ &\quad + 6144\Delta^{-6}x^4 - 2304\Delta^{-7}x^6 + 256\Delta^{-8}x^8,\\ \lambda_5(x,y)|_{y=x} &= \lambda_5(x,x) = 1152\cdot\Delta^{-4} - 2304\cdot\Delta^{-5}\cdot x^2\\ &\quad + 3840\cdot\Delta^{-6}(x^2)^2 - 1536\cdot\Delta^{-7}(x^2)^3 + 256\Delta^{-8}(x^2)4\end{aligned} \tag{8.18}$$

Substituting expressions (8.11), (8.12), (8.15), and (8.16) into (8.10), one gets

$$I_\theta(x^2) = \left[1 + \frac{3}{4}\theta^2\cdot\Delta^{-1}.\Box_x + \frac{5}{32}\theta^4\cdot\Delta^{-2}\cdot\Box_x^2 + \cdots\right] e^{-\frac{x^2}{\Delta}} \tag{8.19}$$

where we have used the identities

$$\Box_x\, e^{-\frac{x^2}{\Delta}} = 8\frac{1}{\Delta}\left(\frac{x^2}{2\Delta} - 1\right) e^{-\frac{x^2}{\Delta}} \tag{8.20}$$

and

$$\Box_x^2\, e^{-\frac{x^2}{\Delta}} = \frac{64}{\Delta^2}\left[\left(\frac{x^2}{2\Delta}-1\right)^2 - \left(\frac{x^2}{2\Delta}-1\right) - \frac{1}{2}\right] e^{-\frac{x^2}{\Delta}} \tag{8.21}$$

According to Eq. (8.19) expression (8.9) acquires the form

$$\theta_\theta\left(l^2 - x^2\right) = \left[1 + \frac{3}{4}\theta^2\Box\frac{d}{dx^2} + \frac{5}{32}\theta^4\Box^2\frac{d^2}{d(x^2)^2} + \cdots\right]\theta\left(l^2 - x^2\right) \qquad (8.22)$$

where

$$\frac{d}{dx^2}\theta\left(l^2 - x^2\right) = -\delta\left(l^2 - x^2\right)$$

Thus, we see that Eq. (8.22) is a generalized function with finite support. Further, making use of a formal link

$$\frac{1}{\Delta} = \frac{1}{x^2}\left(1 + \frac{1}{2}\sqrt{4 + x^2\Box}\right)$$

between the parameter Δ and the D'Alembertian operator $\Box$, arisen from Eq. (8.20) one can write expression (8.22) in an another compact form

$$\theta_\theta\left(l^2 - x^2\right) = K(\theta\Box)\theta\left(l^2 - x^2\right) \qquad (8.23)$$

where

$$K(\theta\Box) = 1 + \frac{3}{4}\theta^2\Box\left[\frac{1}{x^2}\left(1 + \frac{1}{2}\sqrt{4 + x^2\Box}\right)\right] + \frac{5}{32}\theta^4\Box^2\left[\frac{1}{x^2}\left(1 + \frac{1}{2}\sqrt{4 + x^2\Box}\right)\right] + \cdots \qquad (8.24)$$

Similarly, the Dirac δ-function in noncommutative spacetime is also changed

$$\delta^4(x) \Rightarrow \delta^4_\theta(x) = \lim_{\Delta\to 0}\left[\left(\sqrt{1/\Delta\pi}\right)^4 e^{-\frac{x^2}{2\Delta}} \star e^{-\frac{x^2}{2\Delta}}\right] = K(\theta\Box)\delta^4(x) \qquad (8.25)$$

Owing to above formulas in noncommutative spacetime the wave packet located at the origin takes the form

$$W(x, \Delta) = \left(\sqrt{1/\Delta\pi}\right)^4 e^{-\frac{x_0^2 + x^2}{\Delta}} \Rightarrow W_\theta(x, \Delta) = \left(\sqrt{1/\Delta\pi}\right)^4 e^{-\frac{x^2}{2\Delta}} \star e^{-\frac{x^2}{2\Delta}} \qquad (8.26)$$

Here evolution of the wave packet may be understood by means of the proper time formulism(s) instead of the usual time variable ($x_0 = t$). The covariant $\star$-product of this expression is

$$W_\theta(x, \Delta) = \varphi_\theta W(x, \Delta) \qquad (8.27)$$

where

$$\varphi_\theta(x) = 1 + 6\frac{\theta^2}{\Delta^2}P_1 + 10\frac{\theta^4}{\Delta^4}P_2 + \cdots \qquad (8.28)$$

Here

$$P_1 = \frac{x_E^2}{2\Delta} - 1, \qquad P_2 = P_1^2 - P_1 - \frac{1}{2} \tag{8.29}$$

We see that $\varphi_\theta(x)$ is a polynomial.

Consider yet one quantity of interest. This is metric or distance of two events in noncommutative spacetime:

$$\begin{aligned} x_\theta^2 &= \sqrt{x^2}(\star)_c\sqrt{x^2} = \cos\mathrm{h}\,\theta\sqrt{\left(\partial_\mu^x \cdot \partial_y^\mu\right)^2 - \Box_x\Box_y} \cdot \sqrt{x^2}\sqrt{y^2}|_{y=x} \\ &= x^2\left[1 - \frac{6}{2}\cdot\frac{\theta^2}{x^4} + \frac{100}{24}\cdot\frac{\theta^4}{x^8} - \cdots\right] \end{aligned} \tag{8.30}$$

This metric tends to the usual one, $x^2 = -x_0^2 + \mathrm{x}^2$ at long distances.

Finally, notice that the plan wave $\exp(i\omega t - i\mathbf{px}) = \varphi(x)$ possesses remarkable properties. It does not alter its form in noncommuatative spacetime:

$$\varphi_\theta(x) = e^{ipx/2}(\star)_c\, e^{ipx/2} = \Lambda_{xy}\, e^{ipx/2} \cdot e^{ipy/2}|_{y=x} = e^{ipx} \tag{8.31}$$

Commutation relations (1.1) allow us to link the dimensionful parameter θ with the Heisenberg uncertainty relations $\Delta x \cdot \Delta p \sim \hbar$. Indeed, from Eq. (8.27) it follows

$$1 = 1 + 6\frac{\theta^2}{(\Delta x)^2}P_1 + 10\frac{\theta^4}{(\Delta x)^4}P_2 + \cdots \tag{8.32}$$

at the limit $x \to 0$. Here we have changed $\Delta \to \Delta x$, $\sqrt{\Delta x}$ is the width of the wave packet and functions P_1 and P_2 are given by Eq. (8.29), where $P_1(0) = -1$, $P_2(0) = \frac{3}{2}$.

Thus, Eq. (8.32) gives

$$\theta = \sqrt{\frac{6}{15}} \cdot \Delta x \tag{8.33}$$

at least up to θ^6-th order in θ, and therefore

$$1.25\sqrt{\theta} \cdot \Delta\, p \sim \hbar \tag{8.34}$$

This is quantum mechanical physical meaning of the parameter θ for noncommutative spacetime. In this case, the metric x_θ^2 (8.30) is oscillated at small distances, but it is possible that the metric becomes infinite at the origin due to the uncertainty relation (8.34).

8.2. The Class of Test Functions and Generalized Functions in NQFD

We know that initial objects of the local QFT are singular functions:

$$\textit{the causal Green function } \Delta_c(x) \tag{8.35}$$

or the propagator of the particle with mass m.

$$\textit{the positive-frequency part } \Delta_+(x) \tag{8.36}$$

of the Pauli-Jordan function $\Delta(x - y) = [\varphi(x), \varphi(y)]-$, where $\varphi(x)$ is the field operator. These functions have a higher singularity of the type of $\delta(\lambda)$, $\theta(\lambda)$, $\lambda^{-1/2}$ on the light cone $\lambda = -x_0^2 + \mathbf{x}^2 = 0$, and are studied by means of countable normalized spaces (spaces of test functions):

(1) Space D(G). Let G be the finite region, i.e., the bounded open set in the n-dimensional real space $\mathbb{R}^n$. Denote D(G), the set of infinitely smooth functions (i.e., functions having continuous partial derivatives of all orders) in $\mathbb{R}^n$, tending to zero outside the region G. Define in D(G) the countable system norm q_σ by the formula ($u(x) \in$ D(G)):

$$q_0(u) = p_{00}(u) = \sup_x |u(x)|, \ldots, \quad q_\sigma(u) = p_{o\sigma} = \max_{|\alpha|\le\sigma} \sup_x |D^\alpha u(x)|, \tag{8.37}$$

where

$$D^\beta = \frac{\partial^{\beta_1} + \cdots + \beta_n}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \tag{8.38}$$

(2) Space S [or S($\mathbb{R}^n$)] consists of all infinitely smooth functions in the n-dimensional real space $\mathbb{R}^n$, which decrease rapidly any polynomials of $(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}$ together with all partial derivatives at $||x|| \to \infty$, i.e., S = S($\mathbb{R}^n$) = C(∞, ∞, $\mathbb{R}^n$). For these functions all the norms.

$$p_{\rho\sigma} = \cdot \max_{|\alpha|\le\rho, |\beta|\le\sigma} \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)| \tag{8.39}$$

take finite values. Here the space C (σ, ρ, n) consists of complex-valued functions of n-real variables $x = x_1, \ldots, x_n$, having continuous partial derivatives up to the order σ inclusively, and decreasing no more slowly than $|x|^{-\rho}$ together with all derivatives at infinity. In other words, for the functions u(x) $\in$ C(σ, ρ, n) all the products of the type

$$x^\alpha D^\beta u(x) \qquad |\alpha| \le \rho, \quad |\beta| \le \sigma \tag{8.40}$$

are bounded. The norm in the space C(σ, ρ, n) is given by (8.39). We will define the convergence in S by the countable system norms

$$p_\sigma(u) = p_{\sigma\sigma}(u) = \max_{|\alpha|\le\sigma, |\beta|\le\sigma} \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)|, \quad \sigma = 0, 1, 2, \ldots \tag{8.41}$$

In particular, Hermit–Chebyshev functions, and in general, all functions of the type

$$P(x_1, x_2, \ldots, x_n)\ \exp\left(-x_1^2/a_1^2 - \cdots - x_n^2/a_n^2\right) \tag{8.42}$$

may be used as functions of S-space, where P $(\cdot)$ is an arbitrary polynomial.

Definition 8.2. A generalized function is called any linear continuous functional over the countably normalized space S defined above, i.e., any element of space S′. Space S′ consists of all functional of the type

$$F(u) = \int_{\mathbb{R}^n} d^n x\ f(x) D^\alpha u(x) \tag{8.43}$$

where D^α is given by formula (8.38), and $f(x)$ is a continuous function of the polynomial growth. The function of space S are called test functions. The concept of the generalized functions depends on the choice of the initial (linear topological) space of the test functions. For example, instead of S we would take D(G) as a base of test space. Schwartz (1957, 1959) defined the generalized functions as the continuous functionals on space D of all finite and infinite differentiable functions (D is the union of the space D (G), when the region G is changed).

Definition 8.3. Functions disappearing outside some finite region of space are called finite functions. Closure of points set on which a continuous function $u(x) \neq 0$ is called the support of this function.

From (8.27) and (8.28) it follows that with respect to the test functions of generalized functions, spacetime noncommutativity plays a role of multiplier $\varphi_\theta(x)$. It is said that function $\varphi_\theta(x)$ is multiplier in space S for test functions if from $u(x) \in$ S it follows that $\varphi_\theta(x)\,\mathrm{u}(x) \in$ S. The space of all multipliers arase from spacetime noncommutativity we denote Ω_θ. It is clear that if $\varphi(x)$ is infinitely differentiable and a polynomially bounded function of x (together with all its derivatives), then $\varphi(x)$ is the multiplier in S. The functional series (8.28) satisfies this condition.

In the local QFT concept of locality and microcausality condition is connected with a definition of the local properties of the test and generalized functions, and requires among them the existence of functions with bounded support. Usually, as such functions one can consider the functions of D-space of infinite-differentiable functions with bounded supports. By means of these functions the spacetime properties of the functional are investigated, for example, the commutator of Heisenberg's field $\varphi(x)$:

$$[\varphi(x), \varphi(y)]_- = ? \tag{8.44}$$

or the causality condition for the S-matrix in the Bogolubov form

$$\frac{\delta}{\delta g(x)}\left(\frac{\delta S}{\delta g(y)} S^{+}\right) = 0 \tag{8.45}$$

for $x \lesssim y$.

However, in spacetime noncommutativity there appear nonlocal distributions (or generalized functions) of the type of (8.23) and (8.25). Study of such singular functions is needed in introduction of a new class of test functions named entire functions of the finite order of growth α. The space of these functions is denoted Z_α (Efimov, 1977).

For any $f(z_1, \ldots, z_n) \in Z_\alpha$ there exist such positive numbers $C > 0$ and $A_j > 0$ $(j = 1, \ldots, n)$ that

$$|f(z_1, \ldots, z_n)| \leq C \exp\left[\sum_{j=1}^{n} A_j |z_j|^\alpha\right] \tag{8.46}$$

and for any $y_1, \ldots, y_n$

$$\int d^4x_1 \ldots \int d^4x_n \, |f(x_1 + iy_1, \ldots, x_n + iy_n)| < \infty \tag{8.47}$$

The number α is chosen depending on the interaction Lagrangian under consideration and the way of introducing nonlocality into the theory.

As seen below, for study of the $S_\star$-matrix in NQFD constructed by using the covariant $(\star)_c$-product it is sufficient to use D-space of infinite differentiable functions of the type of (8.42) in accordance with the particular example (8.19).

8.3. Structural Peculiarity of the $S_\star$-Matrix in NQFT

As seen above, the $S_\star$-matrix in NQFT is constructed as in the local theory by means of $T_\star$-product of field operators:

$$S_\star = T_\star \exp\{i \int d^4x, \mathcal{L}^\star_{\text{in}}(x)\} \tag{8.48}$$

where the interaction Lagrangian is formed by using the $\star$-product, for example

$$\mathcal{L}^\star_{\text{NQED}} = i\, e\bar{\psi}(x) \star \gamma^\mu \psi(x) \star A_\mu(x) \tag{8.49}$$

and

$$\mathcal{L}^\star_{\phi^3} = \frac{g}{3!}\varphi(x) \star \varphi(x) \star \varphi(x) \tag{8.50}$$

and so on.

Theorem 1. *The Wick theorem is valid for the NQFT with the covariant $(\star)_c$-product.*

The proof is trivial. Indeed, as first step one gets

(1).
$$\langle 0|T[\varphi(x)(\star)_c\varphi(y)]|0\rangle = \Lambda_{xy}\langle 0|T[\varphi(x)\varphi(y)]|0\rangle$$
$$= \Lambda_{xy}\Delta(x-y) = \Delta(x-y) \qquad (8.51)$$

(2). As second step, let us consider simple expression:

$$\begin{aligned} I_{123} &= \langle 0|T\{:\varphi(x_1):(\star)_c:\varphi(x_2):(\star)_c:\varphi(x_3):\}|0\rangle \\ &= \langle 0|T\{\Lambda_{12}\Lambda_{13}:\varphi(x_1)::\varphi(x_2)::\varphi(x_3): \\ &\quad + \Lambda_{13}\Lambda_{32}:\varphi(x_1)::\varphi(x_3)::\varphi(x_2): \\ &\quad + \Lambda_{21}\Lambda_{13}:\varphi(x_2)::\varphi(x_1)::\varphi(x_3): \\ &\quad + \Lambda_{23}\Lambda_{31}:\varphi(x_2)::\varphi(x_3)::\varphi(x_1): \\ &\quad + \Lambda_{31}\Lambda_{13}:\varphi(x_3)::\varphi(x_1)::\varphi(x_3): \\ &\quad + \Lambda_{32}\Lambda_{21}:\varphi(x_3)::\varphi(x_2)::\varphi(x_1):\}|0\rangle \end{aligned} \qquad (8.52)$$

where we have used the short notation:

$$\Lambda_{12}\cos\text{h}\,\theta\sqrt{\left(\partial_\rho^{x_1}\cdot\partial_{x_2}^{\rho}\right)^2 - \Box_{x_1}\Box_{x_2}},$$

and

$$\varphi_1 = \varphi(x_1) \quad \text{and etc.}$$

Exposing T-product in (8.52) we have

$$\begin{aligned} I_{123} &= 2(\Lambda_{12}\Lambda_{13} + \Lambda_{13}\Lambda_{32} + \Lambda_{21}\Lambda_{13})[\Delta_{12}:\varphi_3 \\ &\quad + \Delta_{13}:\varphi_2: + \Delta_{23}:\varphi_1:] \end{aligned} \qquad (8.53)$$

where $\delta_{12} = \Delta(x_1 - x_2)$ is the Green function of the $\varphi(x)$ field. Carry out some simple calculations in (8.53) and obtain

$$\begin{aligned} I_{123} = 2\{&\Lambda_{12}\Lambda_{23}\Delta_{12}:\varphi_3: + \Lambda_{13}\Lambda_{32}\Delta_{12}:\varphi_3: + \Lambda_{21}\Lambda_{13}\Delta_{12}:\varphi_3: \\ &+ \Lambda_{12}\Lambda_{23}\Delta_{13}:\varphi_2: + \Lambda_{13}\Lambda_{32}\Delta_{13}:\varphi_2: + \Lambda_{21}\Lambda_{13}\Delta_{13}:\varphi_2: \\ &+ \Lambda_{12}\Lambda_{23}\Delta_{23}:\varphi_1: + \Lambda_{13}\Lambda_{32}\Delta_{23}:\varphi_1: + \Lambda_{21}\Lambda_{13}\Delta_{23}:\varphi_1:\} \end{aligned}$$

Further, use the identity $\Lambda_{12}\Delta_{12} = \Delta_{12}\ldots$, arisen from (8.51) and obvious equality $\Lambda_{12}\Lambda_{32}:\varphi_3:\Delta_{12} = \Lambda_{12}\Delta_{12}:\varphi_3:$, $\Lambda_{12}\Lambda_{23}:\varphi_2:\Delta_{13} = \Lambda_{13}\Delta_{13}:\varphi_2:$ and etc., and go back to the $\star$-product, we obtain the almost local expression

$$I_{123} = 6\{\Delta_{12}\star:\varphi_3: + \Delta_{13}\star:\varphi_2: + \Delta_{23}\star:\varphi_1:\} \qquad (8.54)$$

with only difference in the verex points with the $\star$-product (Fig. 7).

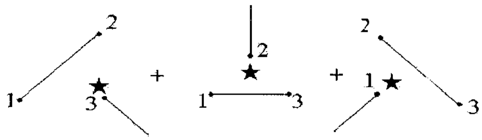


Fig. 7. The Simpler Feynman diagram arisen from T_*-product in NQFT.

Now we consider yet one $T_\star$-product for the ϕ^3-theory.

$$I_3 = \langle 0|T\{: \varphi_1 \star \varphi_1 \star \varphi_1 :: \varphi_2 \star \varphi_2 \star \varphi_2 :: \varphi_3 \star \varphi_3 \star \varphi_3\}|0\rangle$$
$$= \prod_{\substack{\text{permutations}\\ \text{of } y,\cdots,y,}} \{\Lambda_{y_1y_2}\Lambda_{y_2y_3}\Lambda_{y_3y_4}\Lambda_{y_4y_5}\Lambda_{y_5y_6}\Lambda_{y_6y_7}\Lambda_{y_7y_8}\Lambda_{y_8y_9}\}$$
$$\times\langle 0|T\{: \varphi(y_1)\varphi(y_2)\varphi(y_3) : |_{y_1=y_2=y_3=x_1} : \varphi(y_4)\varphi(y_5)\varphi(y_6) : |_{y_4=y_5=y_6=x_2}$$
$$\times : \varphi(y_7)\varphi(y_8)\varphi(y_9) : |_{y_7=y_8=y_9=x_3}\}|0\rangle \quad (8.55)$$

One of terms with the following chronological pairing

$$: \varphi\,(y_1)\varphi\,\underbrace{(y_2)\varphi(y_3) :: \varphi}(y_4)\,\overbrace{\varphi(y_5)\varphi(y_6) :: \varphi}(Y_7)\varphi(y_8)\varphi(y_9) :$$

gives the matrix element (Fig. 8)

$$I_3^1 =: \varphi_{x_1} : \star\Delta(x_1 - x_2)\star : \varphi_{x_2} : \star\Delta(x_2 - x_3)\star : \varphi_{x_3} : \star\Delta(x_3 - x_1) \quad (8.56)$$

after going back to the star-product.

In the momentum space matrix element corresponding to the diagram shown in Fig. 8. gives the $\star$-product of the type:

$$e^{-ipx_1} \star e^{-i(p-k)(x_1-x_3)} \star e^{-i(p'-p)x_3} \times e^{-i(p'-k)(x_3-x_2)} \star e^{-ik(x_2-x_1)} \star e^{ip'x_2} \quad (8.57)$$

as it will be expected (see Eq. (7.1)).

Thus, the $S_\star$-matrix elements in the noncommutative quantum field theory are constructed by a similar way as in the usual local theory with using the star product.

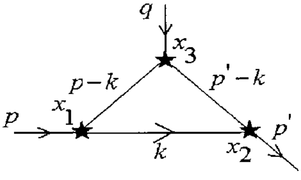


Fig. 8. One of the diagrams in the noncommutative ϕ^3-theory.

8.4. The Proof of Unitarity and Causality of the $S_\star$-Matrix

8.4.1. The Causality Condition in the Functional Form

We now verify the condition (8.45) for the $S_\star$-matrix, which is constructed by means of the $\star$-product of the interaction Lagrangian $\mathcal{L}^\star_{in}(x)$ in the case of the NQFT. As usual, the $S_\star$-matrix is presented as a functional series over the powers of the coupling constant $g(x)$, made into a function of spacetime:

$$S_\star[g] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots \int d^4x_n S^\star_n(x_1, \dots, x_n) g(x_1) \star \dots \star g(x_n) \quad (8.58)$$

or in the convenient form

$$S_\star[g] = T_\star \exp\left\{ i \int d^4x \mathcal{L}^\star_{in}(x) \star g(x) \right\} \quad (8.59)$$

To check condition (8.45), we calculate the variational differentil of $S_\star[g]$ at the point y

$$-i \frac{\delta}{\delta g(y)} (\star)_c S_\star[g] = T_\star \left\{ \mathcal{L}^\star_{in}(y)(\star)_c \exp\left[i \int d^4x \mathcal{L}^\star_{in}(x)(\star)_c g(x) \right] \right\}$$

taking into account here $\delta g(x) \star / \delta g(y) = \delta^4(x-y)$ in accordance with the formula (8.7).

Further, the four-dimensional space $\mathbb{R}^4$ is divided into two parts G_+ and G_- by the space-like surface $x_0 = \text{const} = y_0$ with respect to which G_+ lies to "the future" and G_- to "the past." Thus, we have

$$\begin{aligned} -i \frac{\delta}{\delta g(y)} (\star)_c S_\star[g] &= T_\star \left\{ \mathcal{L}^\star_{in}(y)(\star)_c \exp\left[i \int_{G_+} d^4x \mathcal{L}^\star_{in}(x)(\star)_c g(x) \right] \right. \\ &\quad \left. \times (\star)_c \exp\left[i \int_{G_-} d^4z \mathcal{L}^\star_{in}(z)(\star)_c g(z) \right] \right\} \\ &= T_\star \left\{ \mathcal{L}^\star_{in}(y)(\star)_c \exp\left[i \int_{G_+} d^4x \mathcal{L}^\star_{in}(x)(\star)_c g(x) \right] \right\} \\ &\quad \times (\star)_c T_\star \left\{ \exp\left[i \int_{G_-} d^4z \mathcal{L}^\star_{in}(z)(\star)_c g(x) \right] \right\} \end{aligned} \quad (8.60)$$

On the other hand, we reach by analogy

$$\begin{aligned} S_\star[g] &= T_\star \left\{ \exp\left[i \int_{G_+} d^4x \mathcal{L}^\star_{in}(x)(\star)_c g(x) + i \int_{G_-} d^4z \mathcal{L}^\star_{in}(z)(\star)_c g(z) \right] \right\} \\ &= T_\star \left\{ \exp\left[i \int_{G_+} d^4x \mathcal{L}^\star_{in}(x)(\star)_c g(x) \right] \right\} T_\star \left\{ \exp\left[i \int_{G_-} d^4z \mathcal{L}^\star_{in}(z) \star g(z) \right] \right\} \end{aligned}$$

and also

$$S_\star^+[g] = \left\{ T_\star \exp\left[i \int_{G_-} d^4x \mathcal{L}_{in}^\star(x)(\star)_c g(x) \right] \right\}^+$$
$$\times \left\{ T_\star \exp\left[i \int_{G_+} d^4z \mathcal{L}_{in}^\star(z)(\star)_c g(z) \right] \right\}^+$$

From this, taking into account the unitarity property for

$$T_\star \left\{ \exp\left[i \int_{G_-} d^4x \mathcal{L}_{in}^\star(x)(\star)_c g(z) \right] \right\}$$

and making use of (8.60), we get

$$-i \frac{\delta}{\delta g(y)}(\star)_c S_\star^+[g] = T_\star \left\{ \mathcal{L}_{in}^\star(y)(\star)_c \exp\left[i \int_{G_-} d^4z \mathcal{L}_{in}^\star(z)(\star)_c g(z) \right] \right\}$$
$$\times \left\{ T_\star \exp\left[i \int_{G+} d^4z \mathcal{L}_{in}^\star(z)(\star)_c g(z) \right] \right\}^+$$

Therefore, the product

$$\left[\frac{\delta}{\delta g(y)}(\star)_c S_\star[g] \right] S_\star^+[g]$$

does not depend on the behavior of the function $g(x)$ in the region G_-, i.e., for $x_0 < y_0$. In accordance with the covariant principle of the relativistic NQFT with using the covariant $(\star)_c$-product it takes place also in the case when $x \sim y$ (space-like separation). We recall that in this region ($x \sim y$) the commutator of the covariant $(\star)_c$-product field operator $\varphi(x)$ (in particular for the scalar theory):

$$[\varphi(x), (\star)_c \varphi(y)]_- = \Delta(x-y)$$

disappears, which ensures the independence of events separated by space-like intervals, i.e., the causality condition in noncommutative spacetime with the covariant $(\star)_c$-product.

All above statements are based on the formal functional method. However, there exists the perturbation theory approach (or the diagrammar approach) to investigate unitarity and causality conditions in each order of the perturbation series (for example, see 't Hooft and Veltman, 1973).

8.4.2. Diagrammar Approach to the Study of Causality and Unitarity Conditions in NQFT

As seen above, structural aspects of Feynman diagrams in NQFT are very similar to ones in the local QFT with only difference in those vertices, and therefore one can study causality and unitarity conditions by means of diagrammers. Sketches of study of such problem are divided into several steps:

8.4.2.1. The Kallén–Lehmann Representation.

$$f(-s) = \int_{a\geq 0}^{\infty} \frac{\rho(s')}{s'-s-i\varepsilon} ds' \tag{8.61}$$

is valid for any propagator, for instance, for vector mesons

$$i(2\pi)^4 \tilde{\Delta}_{\mu\nu}(k) = \delta_{\mu\nu} f_1(k^2) + k_\mu k_\nu f_2(k^2) \tag{8.62}$$

In (8.61) the functions $\rho(s')$ must be real. The statement that any function $\Delta(x)$ satisfies the Kallén–Lehmann representation is equivalent to the statement

$$\Delta(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x) \tag{8.63}$$

where $\Delta^+(\Delta^-)$ is a positive (negative) energy function

$$\Delta^{\pm}(x) = \frac{1}{(2\pi)^3} \int_{a\geq 0}^{\infty} ds' \, \rho(s') \int d^4k \, e^{ikx} \theta(\pm k_0)\delta(k^2+s') \tag{8.64}$$

From this equation it follows that $\delta^+(x_0) = \Delta^-(-x_0)$, so in any case $\Delta^+(0) = \Delta^-(0)$. This is enough to treat the case of one derivative such as occurring in the case of fermion propagators. It is obvious that $\partial_0\Delta^+ = \partial_0\Delta^-$ in $x_0 = 0$ only if the dispersion integral is supercovergent

$$\int ds' \, \rho(s') = 0 \tag{8.65}$$

Let us now assume that the superconvergence Eq. (8.65) holds. Then one gets

$$\partial_\mu\partial_\nu[\theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x)] = \theta(x_0)\partial_\mu\partial_\nu\Delta^+(x) + \theta(-x_0)\partial_\mu\partial_\nu\Delta^-(x) \tag{8.66}$$

using

$$\partial(x_0)\partial_0\Delta^+(x) - \delta(x_0)\partial_0\Delta^-(x) = 0,$$

as well as

$$\delta'(x_0)\Delta^+(x) - \delta'(x_0)\Delta^-(x) = 0.$$

Equation (8.66) is crucial for the proof of unitarity and causality in $S_\star$-matrix theory in accordance with the formula (8.51).

8.4.2.2. Unitary Regulators. Since Green functions (8.51) in NQFT as well as in the local QFT are divergent, it is needed to introduce so-called unitarity regulators, say

$$\tilde{\Delta}(k) \to \tilde{\Delta}_\delta(k) = \frac{1}{(2\pi)^4 i} \frac{(1/\delta)^2}{(k^2+m^2)(k^2+m^2+\delta^{-2})} \tag{8.67}$$

Such type of the regulator method works for any theory in the sense that it allows a proper definition of all diagrams and is moreover very suitable in the connection with the proof of unitarity and causality. However, it fails in the case of Lagrangians invariant under a gauge group, for which it will introduce a more sophisticated method.

8.4.2.3. Method of Cutting Equations. It is supposed that diagrams are sufficiently regularized (say, the intermediate regularization like δ (8.67)), so that no divergencies occur. The propagator of a particle is divided into positive and negative energy (frequency) parts

$$\Delta_{ij}(x) = \theta(x_0)\Delta^+_{ij}(x) + \theta(-x_0)\Delta^-_{ij}(x) \tag{8.68}$$

$$\Delta^\pm_{ij}(x) = (2\pi)^{-3} \int d^4k\, e^{ikx}\theta(\pm k_0)\rho(k^2) \tag{8.69}$$

with $x = x_i - x_j$, and $\Delta_{ij}(x) = \Delta_{Fij}(x_i - x_j)$. Owing to the reality of the spectral functions ρ in (8.61), we have $\Delta^\pm_{ij} = (\Delta^\mp_{ij})^*$, also $\Delta^\pm_{ij} = (\Delta^\mp_{ji})$, and therefore

$$\Delta^*_{ij} = \theta(x_i - x_j)\Delta^-_{ij} + \theta(x_j - x_i)\Delta^+_{ij} \tag{8.70a}$$

As before

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau\, \frac{e^{i\tau x}}{\tau - i\varepsilon} = \begin{cases} 1 & \text{if} \quad x_0 > 0 \\ 0 & \text{if} \quad x_0 < 0 \end{cases} \tag{8.70b}$$

and

$$\theta(x) + \theta(-x) = 1. \tag{8.70}$$

Let us consider a diagram with n vertices. As in the local theory such a diagram represents in coordinate space an expression containing many propagators depending on arguments $x_1, \ldots, x_n$. We will denote such an expression by

$$F^\star_\theta(x_1, \ldots, x_n). \tag{8.71}$$

For example, the triangle diagram represents the function (Fig. 9):

$$F^\star_\theta(x_1, x_2, x_3) = (ig)^3 \star \Delta_{31} \star \Delta_{23} \star \Delta_{12}\star \tag{8.72}$$

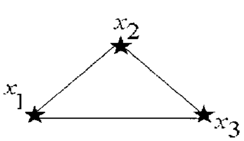


Fig. 9.

Every diagram, when multiplied by the appropriate plane waves (or source functions) with the $\star$-product and integrated over all x, contributes to the $S_\star$-matrix. The contribution to the $S_\star$-matrix, defined by

$$S_\star = 1 + iT_\star \tag{8.73}$$

is obtained by multiplying by a factor $-i$. Unitarity of the $S_\star$-matrix implies an equation for the imaginary part of the so-defined $T_\star$ matrix

$$T_\star - T_\star^+ = iT_\star^+ \star T_\star \tag{8.74}$$

The $T_\star$-matrix, or rather the diagrams, are also constrained by the requirement of causality. Causality is formulated by using proposal involving the off-mass-shell Green's functions $\Delta_\theta^c(x) = \Delta^c(x)$. The causality requirment is most suitable in connection with a diagrammatic analysis. In the language of diagrams Bogolubov's causality condition can be put as follows: if a spacetime point x_1 is in the future with respect to some other space-time point x_2, then the diagrams involving x_1 and x_2 can be rewritten in terms of functions that involve positive energy flow from x_2 to x_1 only.

The difficulty of this defintions is connected with the fact that space-time points cannot be accurately pinpointed with relativistic wave packets corresponding to particles on mass-shell. Therefore, this definition cannot be formulated as an $S_\star$-matrix constraint. It can only be used for Green's functions. By this reason in both commutative and noncommutative QFTs with the covariant $\star$-product the proof of unitarity and causality conditions for the $S_\star$-matrix is the same, since Green's functions in these theories coincide exactly.

There exist other definitions which refer to the properties of the operator fields. In particular there is the proposal of Lehmann *et al.* (1955, 1957) that the fields commute outside the light cone. However, definition of the light cone is changed in NQFT. The formulation of Bogolubov causality in terms of cutting rules for diagrams was done by 't Hooft and Veltman (1973). We will give here the main idea of their scheme of the construction.

8.4.2.4. The Largest Time Equation. Instead of a function (8.71) corresponding to some diagram, let us define new functions $F_\theta^\star$,

$$F_\theta^\star(x_1, x_2, \ldots, \underline{x_i}, \ldots, \underline{x_j}, \ldots, x_n) \tag{8.75}$$

where one or more of the variables $x_1, \ldots, x_n$ are underlined. This function is derived from the original function (8.71) by the following.

1) A propagator Δ_{ki} is unchanged if neither x_k nor x_i is underlined.
2) A propagator Δ_{ki} is replaced by Δ_{ki}^+ if x_k but not x_i is underlined.
3) Δ_{ki} is replaced by Δ_{ki}^- if x_i but not x_k is underlined. (8.76)
4) Δ_{ki} is replaced by Δ_{ki}^* if x_k and x_i are underlined.

5) For any underlined x, replace one factor i by $-i$. Apart from that, the rules for the vertices remain unchanged. This latter fact is very important for NQFT.

Equations (8.68) and (8.70) lead trivially to an important equation, the *largest time equation*. Assume the time x_{i0} is larger than any other time component. Then any function $F^\star_\theta$ in which x_i is not underlined equals minus the same function but with x_i now underlined

$$F^\star_\theta(x_1, x_2, \ldots, x_i, \ldots \underline{x_j}, \ldots, x_n) = -F^\star_\theta(x_1, x_2, \ldots, \underline{x_i}, \ldots \underline{x_j}, \ldots, x_n) \tag{8.77}$$

The minus sign is a consequence of point 5. It is useful to invent a diagrammatic representation of the newly defined functions: Any function $F^\star_\theta$ is represented by a diagram where any vertex corresponding to an underlined variable is provided with a $\star$ circle ($\star$).

Notice that if $F^\star_\theta(x_1, x_2, x_3)$ is given by Eq. (8.72) then

$$F^\star_\theta(x_1, \underline{x_2}, \underline{x_3}) = (ig)^3 \star \Delta^+_{31} \star \Delta^\star_{23} \star \Delta^-_{12} \star \tag{8.78}$$

The corresponding diagram is as follows (Fig. 10):

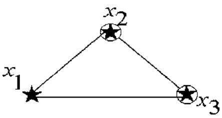


Fig. 10.

If the time component of x_3 is largest we have, for instance (Fig. 11),

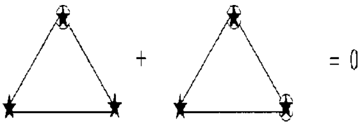


Fig. 11.

For such a diagram it is impossible to see if a given line connecting a star circled to an unstar circled vertex corresponds to a Δ^+ or Δ^- function. But because of Eq. (8.69) the result is the same anyway. Energy always flows from the uncircled to the circled vertex, because of the θ function in Eq. (8.69). Of course there is no restriction on the sign of energy flow for lines connecting two circled or two uncircled vertices.

8.4.2.5. Absorptive Part. To define the contribution of a diagram to the $S^\star$-matrix the corresponding function $F^\star_\theta(x_1, \ldots, x_n)$ must be multiplied with the appropriate plane waves (or source functions) using the $\star$-product for the ingoing and

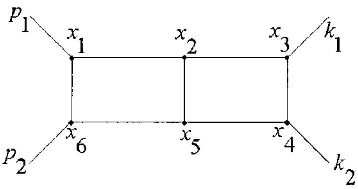


Fig. 12.

outgoing lines and integrated over all x_i. For instance, the function $F^\star_\theta(x_1, \ldots, x_6)$ corresponding to the diagram (Fig. 12) must be multiplied by

$$e^{ip_1x_1}\, e^{ip_2x_6}\, e^{-ik_1x_3}\, e^{-ik_2x_4}$$

with the star product in the appropriate places between the Green functions, and subsequently integrated over $x_1, \ldots, x_6$. The result reads

$$\begin{aligned} S^6_\star = {} & \frac{1}{(2\pi)^{4\cdot 6}} \int d^4x_1 \ldots \int d^4x_6\, e^{ip_1x_1} \star \Delta(x_1 - x_2) \star \Delta(x_2 - x_5) \star \Delta(x_2 - x_3) \\ & \times \star\, e^{-ik_1x_3} \star \Delta(x_3 - x_4) \star e^{-ik_2x_4} \star \Delta(x_4 - x_5) \star \Delta(x_5 - x_6) \\ & \times \star\, e^{ip_2x_6} \star \Delta(x_6 - x_1) \end{aligned} \tag{8.79}$$

Next it will be taken as the covariant star product $(\star)_c$ instead of the usual star-product in (8.79). For the time ordering of the various x_i Eq. (8.77) takes the general form

$$\sum_{\text{inderlinings}} F^\star_\theta(x_1, \ldots, \underline{x_i}, \ldots, \underline{x_j}, \ldots, x_n) = 0 \tag{8.80}$$

The summation is taken over all possible ways that the variables may be underlined. There is also one term, the last, where all avriables are underlined. In this case,

$$F^\star_\theta(\underline{x_1}, \underline{x_2}, \ldots, \underline{x_n}) = F^\star_\theta(x_1, x_2, \ldots, x_n)^* \tag{8.81}$$

The proof of Eq. (8.80) is trivial.

In the momentum space Eq. (8.80) reduces to

$$\tilde{F}_\theta(k_1, \ldots, k_n) + \hat{\tilde{F}}_\theta(k_1, \ldots, k_n) = -\sum_{\text{cuttings}} \tilde{F}^c_\theta(k_1, \ldots, k_n) \tag{8.82}$$

where $\tilde{F}_\theta$ is the Fourier transform of the function F without underlinings, $\hat{\tilde{F}}_\theta$ the Fourier transform of the function F_θ with all variables underlined. The functions F^c_θ correspond to all nonzero diagrams containing both star circled and uncircled (with star) vertices. They correspond to all possible cuttings of the original diagram with the prescription that for a cut line the propagator function $\Delta(k)$ must be replaced by $\Delta^\pm(k)$ with the sign such that energy is forced to flow towards the shaded region. Equation (8.82) is Cutkosky's (1960) cutting rule (for detail, see 't Hooft and Veltman, 1973).

Notice that the $T_\star$-matrix is obtained by multiplying by $-i$, we see that Eq. (8.82) is of precisely the same structure as the unitarity Eq. (8.74). However, Eq. (8.82) holds for a single diagram, while unitarity is a property true for a transition amplitude, that is for the sum of diagrams contributing to a given process.

Equation (8.82) holds for any theory described by a Lagrangian, whether it is unitary or not. The Feynman rules for $\hat{\tilde{F}}_\theta$ are, however, different from those for $T_\star^+$. Therefore, if Eq. (8.82) is to truly imply unitarity a number of properties must hold.

8.4.2.6. Causality. Let us consider some diagram that represents a function $F_\theta^\star(x_1, \ldots, x_n)$. Let x_i and x_j be any two variables, and the time component of x_j be larger than x_{i0}. The following equation holds independently of the time ordering of the other time components

$$\sum_{\substack{\text{underlinings}\\ \text{expect } x_i}} F_\theta^\star(x_1, \ldots, \underline{x_k}, \ldots, x_n) = 0 \quad \text{if} \quad x_{i0} < x_{j0} \tag{8.83}$$

Again terms cancel in parts. We do not need the diagrams where x_i is underlined, because x_{i0} is never the largest time.

Equation (8.83), when multiplied by the appropriate source (or plane wave) functions and integrated over all x except x_i and x_j, is the single diagram version of Bogolubov's causality condition. His notation is

$$\left[\frac{\delta}{\delta g(x_i)}(\star)_c \frac{\delta}{\delta g(x_j)}(\star)_c S^\star\right](\star)_c \hat{S}_\star$$
$$+ \frac{\delta}{\delta g(x_i)}(\star)_c S_\star (\star)_c \frac{\delta}{\delta g(x_j)}(\star)_c \hat{S}_\star = 0 \quad \text{if} \quad x_{i0} < x_{j0}. \tag{8.84}$$

Here the first term describes cut diagrams (including the case of no cut at all—the unit part of $\hat{S}_\star$) with x_i and x_j not circled, and the second term denotes diagrams with x_j but not x_i circled (star). $\hat{S}_\star$ is the $S_\star$-matrix obtained from the conjugate Feynman rules (i.e., all wertices underlined), and will often be equal to S^+. Further, as before $g(x)$ is the coupling constant, made into a function of spacetime.

Similarly we consider the case when $x_{i0} > x_{j0}$. Then we have an equation where now x_j is never to be underlined. Separating off the term with no variable underlined, one can combine equation, with the result

$$F_\theta^\star(x_1, \ldots, x_n) = -\,\theta(x_{j0} - x_{i0}) \sum_i{}' F_\theta^\star(x_1, \ldots, \underline{x_k}, \ldots, x_n)$$
$$-\,\theta(x_{i0} - x_{j0}) \sum_j{}' F_\theta^\star(x_1, \ldots, \underline{x_k}, \ldots, x_n) \tag{8.85}$$

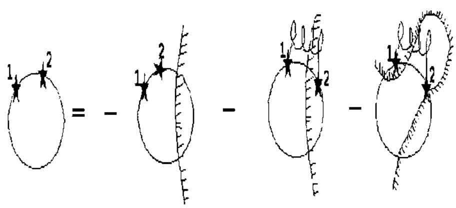
The prime indicates absence of the term without underlined variables. The index i implies absence of diagrams with x_i underlined.

The summations in Eq. (8.85) still contain many identical terms, namely those where neither x_i or x_j is underlined. All these may be taken together to give

$$F_\theta^\star(x_1,\ldots,x_n) = -\sum_{ij}{}' F_\theta^\star(x_1,\ldots,x_n) - \theta(x_{j0}-x_{i0}) \sum_{\substack{\text{underlined}\\ i \text{ not}}} F_\theta^\star(x_1,\ldots,x_n)$$
$$- \theta(x_{i0}-x_{j0}) \sum_{\substack{\text{underlined}\\ i \text{ not}}} F_\theta^\star(x_1,\ldots,x_n) \qquad (8.86)$$

The first term on the right-hand side of Eq. (8.86) is a set of cut diagrams, with x_i and x_j always in the unshaded region. They represent the product $S_\star(\star)_c \hat{S}_\star$ with the restriction that x_i and x_j are vertices of $S_\star$. One can apply in this covariant star product. Doing this as many times as necessary, the right-hand side of Eq. (8.86) can be reduced entirely to the sum of two terms, one containing a function $\theta(x_{i0} - x_{j0})$ multiplying a function whose Fourier transforms contains θ-functions forcing energy flow from i to j, the other containing the opposite combination. This is precisely of the form indicated in Section 8.4.2.3.

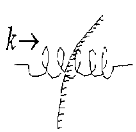
Now turn to Eq. (8.86). Introducing for $\theta(x)$ the Fourier representation Eq. (8.70*b*), one can see that θ as another kind of propagator connecting the points x_i and x_j. Multiplying by the appropriate source (plane wave) functions and integrating over all x_i, we obtain the following diagrammatic equation (Fig. 13):



(8.87)

Fig. 13. The blob stands for any diagram or collection of diagrams.

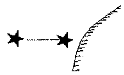
The points 1 and 2 indicate two arbitrary selected vertices. The "self-inductance" is the correction due to the θ-function, and is obviously noncovariant:

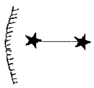


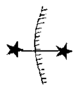
$$= \frac{1}{(2\pi)i}\,\frac{1}{-k_0 - i\varepsilon}\,\delta^3(\vec{k}) \qquad (8.88)$$

In the diagrams (8.87) on the right-hand side summation over all cuts with the points 1 and 2 in the position shown is intended.

The Feynman rules for the cut diagrams (for the simple scalar theory):


Propagator in unshadowed region: $\frac{1}{(2\pi)^4 i}\frac{1}{k^2+m^2-i\varepsilon}$

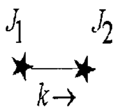

Propagator in shadowed region: $-\frac{1}{(2\pi)^4 i}\frac{1}{k^2+m^2+i\varepsilon}$ (8.89)


Cutline $\frac{1}{(2\pi)^3}\theta(k_0)\delta(k^2+m^2)$

Vertex in unshadowed region: $ig(2\pi)^4$.
Vertex in shadowed region: $-ig(2\pi)^4$.

For a spin-1/2 particle everything obtains above by multiplying with the factor $-i\hat{k}+m$.

Let us consider expression



$(2\pi)^4 i\, J_1(k)\frac{1}{k^2+m^2-i\varepsilon} J_2(k)$

with J_1 and J_2 nonzero if $k_0 > 0$. The unitarity Eq. (8.82) reads (Fig. 14),

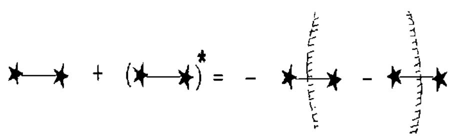


Fig. 14.

The complex conjugation does apply to everything except the sources J. The second term on the right-hand side is zero, because of the condition $k_0 > 0$. The equation becomes

$$J\left[i(2\pi)^4\frac{1}{k^2+m^2-i\varepsilon} - i(2\pi)^4\frac{1}{k^2+m^2+i\varepsilon}\right]$$
$$= J\left[\frac{i^2(2\pi)^8}{(2\pi)^3}\theta(k_0)\delta(k^2+m^2)\right]J$$

Note that the vertex in the shadowed region gives a factor $-i(2\pi)^4$. With

$$\frac{1}{a-i\varepsilon} = P\left(\frac{1}{a}\right) + i\pi\delta(a),$$

it is seen that the equation holds true. Also Eq. (8.87) can be verified (Fig. 15).

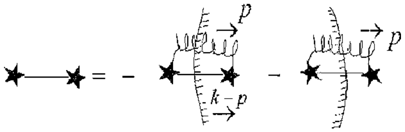


Fig. 15.

We now obtain (note the minus sign for vertex in shadowed region)

$$i(2\pi)^4 \frac{1}{k^2+m^2-i\varepsilon} = \frac{i^2(2\pi)^8}{(2\pi)^3 2\pi i}\int_{-\infty}^{\infty} dp_0 \left\{ \frac{1}{-p_0-i\varepsilon}\theta(k_0-p_0) \right.$$

$$\left. \times\, \delta[(k-p)^2+m^2] + \frac{1}{p_0-i_\varepsilon}\theta(-k_0+p_0)\delta[(k-p)^2+m^2] \right\} |_{p=0}$$

The four-vector p_0 has zero space components (see Eq. (8.88)). The p_0 integration is trivial and gives the desired result.

8.4.2.7. Unitarity. If the cutting Eq. (8.82), diagrammatically represented as (Fig. 16), corresponding to $T_\star - T_\star^+ = iT_\star^+(\star)_c T_\star$, is to imply unitarity, the following must hold:

1) The diagrams in the shadowed region must be those that occur in $S_\star^+$;
2) the Δ^+ functions must be equal to what is obtained when summing over intermediate states.

Notice that point 1 will be true if the Lagrangian generating the $S_\star$-matrix is its own conjugate. Point 2 amounts to the following. The 2-point Green's function, on which the definition of the $S_\star$-matrix source was based, contained a matrix K_{ij}. Indeed, consider the diagrams connecting two sources:
The corresponding expression is

$$\tilde{J}_i(k')G_{ij}(k,k')J_j(k) \tag{8.90}$$

The 2-point Green's function will in general have a pole at some value $-M^2$ of the squared four-momentum k_ν. If there is no pole, there will be no corresponding

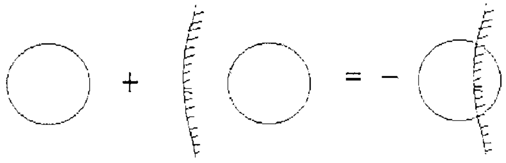
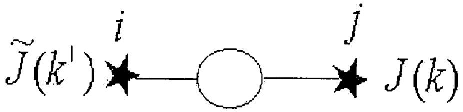


Fig. 16.



$S_\star$-matrix element; such will be the case if a particle becomes unstable because of the interactions. At the pole Green's function will be of the form

$$G_{ij}(k,k') = (2\pi)^4 i\delta^4(k+k')\frac{K_{ij}(k)}{k^2+M^2} \quad \text{at} \quad k^2 = -M^2$$

The matrix residue K_{ij} can be a function of the components k_ν, with the restriction that $k^2 = -M^2$.

One can treat the currents for emission of a particle, corresponding to incoming particles of the $S_\star$-matrix. Define a new set of currents $J_i^{(a)}$ one for every non-zero eigenvalue of K, which are mutually orthogonal and eigenstates of the matrix $K(k)$

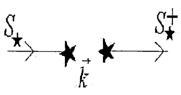
$$J_i^{(a)} J_i^{(b)} = 0 \quad \text{if} \quad a \neq b$$
$$K_{ij}(k) J_j^{(a)}(k) = f^a(k) J_i^{(a)}(k) \tag{8.91}$$

and normalized such that

$$\left[J_j^{(a)}(k)\right]^* K_{ji}(k) J_j^{(a)}(k) = \begin{cases} 1 \text{ for integer spin} \\ \frac{k_0}{m} \text{ for half—integer spin} \end{cases} \tag{8.92}$$

This is possible only if all eigenvalues of K are positive. In the case of negative eigenvalues, normalization is done with minus the right-hand side of Eq. (8.92). The sources thus defined are the properly normalized sources for emission of a particle or an antiparticle (the latter follows from considering $\tilde{K}(-k)$).

Thus in considering $S_\star^+(\star)_c S_\star$ one will encounter (particle-out of $S_\star$, particlein of $S_\star^+$):



$$\sum_a K_{ij}^+(-k) J_j^{*(a)}(k) J_l^a(k) K_{lm}(-k) \tag{8.93}$$

in the sum over intermediate states. K is from the propagators attached to the sources. Because of $\mathcal{L} = (\mathcal{L})_{\text{conjugate}}$ we have $K_{lm}^+(-k) = K_{lm}(k)$. Also if $J(k)K(-k) \sim J(k)$ then $K^+(-k)J^* \sim J^*$, showing that J and J^* are the appropriate eigen currents of $S_\star$ and $S_\star^+$. If unitarity is to be true, we require that this sum (8.93) occurring in $S_\star^+(\star)_c S_\star$ equals the matrix K_{im} occurring when cutting a propagator.

The proof of this is simple. Suppose K_{ij} is diagonal with diagonal elements λ_i. The current-defining Eqs. (8.91) and (8.92) imply that the currents are of the form

$$J^{(a)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1/\sqrt{\lambda_a} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

There are no currents corresponding to zero eigenvalues Obviously,

$$\sum_a J^{(a)*} J^{(a)} = k^{-1} \tag{8.94}$$

and this remains true if one provides the currents with phase factors, etc.

As in the local QFT for spin-1/2 particles things are slightly more complicated, because of γ^4 manipulations. For instance, one will have

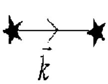
$$K^+(-k)\gamma^4 = \gamma^4 K(k) \tag{8.95}$$

Also the normalization of the currents is different. One finds the correct expression when summing up particle-out/particle-in states, but a minus sign extra for antiparticle-out/antiparticle-in states. This factor is found back in the prescription -1 for every fermion loop.

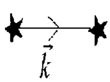
Let

$$\mathcal{L}^\star = -\bar{\psi}(x) \star (\hat{\partial} + m)\psi(x) + \frac{1}{2}\phi(x) \star (\Box - m^2)\phi(x) + g\bar{\psi}(x) \star \psi(x) \star \phi(x)$$

be total Lagrangina of the scalar–spinor interacting system. Then there are four 2-point Green's functions:

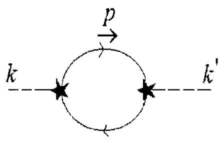


$$\bar{u}^a(k)\frac{-i\hat{k}+m}{k^2+m^2}u^a(k)\frac{2k_0}{4m^2}; \quad a=1,2$$



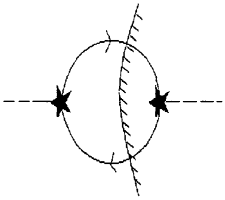
$$-\bar{u}^a(k)\frac{i\hat{k}+m}{k^2+m^2}u^a(k)\frac{2k_0}{4m^2}; \quad a=3,4$$

Note the minus sign for the incoming antiparticle wave function. Scalar particle self-energy is



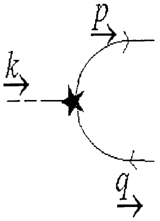
$$-g^2\delta^4(k-k')\int d^4p\cosh\theta\left(\sqrt{(pk)^2-p^2k^2}\right)$$
$$\times\frac{-i\hat{p}+m}{p^2+m^2-i\varepsilon}\frac{-i(\hat{p}-\hat{k})+m}{(p-k)^2+m^2-i\varepsilon}$$

where θ is the dimensionful parameter in spacetime noncommutativity. Note the minus sign for the closed fermion loop. Cut diagram (remember $-i(2\pi)^4$ for vertex in shadowed region):



$$-(2\pi)^2g^2\delta^4(k-k')\int d^4p\cosh\theta\sqrt{(pk)^2-p^2k^2}(-i\hat{p}+m)\theta(p_0)$$
$$\times\delta(p^2+m^2)[-i(\hat{p}-\hat{k})+m]\theta(k_0-p_0)\delta[(p-k)^2+m^2]$$

Decay of scalar into two fermions:



$$ig(2\pi)^4\sqrt{4p_0q_0}\bar{u}(p)u^\alpha(q)\left[\cosh\theta\sqrt{(pk)^2-p^2k^2}\right]^{1/2}\times\delta^4(k-p-q)$$

The superscript α now indicates antiparticle spinor. The complex conjugate, but with k' instead of k, is

$$-ig(2\pi)^4\sqrt{4p_0q_0}\bar{u}^{\alpha}(q)u(p)\left[\cosh\theta\sqrt{(pk')^2-k'^2p^2}\right]^{1/2}\delta^4(k'-p-q)$$

The product of the two summed over intermediate states is

$$\begin{aligned}&(2\pi)^8g^24p_0q_0\int\int\frac{d^3p\,d^3q}{(2\pi)^6 2p_0 2q_0}\left[\cosh\theta\sqrt{(pk)^2-k^2p^2}\right]^{1/2}\\&\times\left[\cosh\theta\sqrt{(pk')^2-k'^2p^2}\right]^{1/2}\times\ \delta^4(k-p-q)\delta^4(k'-p-q)\frac{1}{2p_0}\\&\times(-i\hat{p}+m)\frac{-1}{2q_0}(i\hat{q}+m)\end{aligned}$$

Note the minus sign for the q-spinor sum.

Sing $p_0=\sqrt{\mathbf{p}^2+m^2}$, we have

$$\int\frac{d^3p}{2p_0}=\int d^4p\theta(p_0)\delta(p^2+m^2)$$

and similarly for q. The q integration can be performed

$$\begin{aligned}&-(2\pi)^2g^2\delta^4(k-k')\int d^4p\theta(p_0)\delta(p^2+m^2)\cosh\theta\sqrt{(pk)^2-k^2p^2}\theta(k_0-p_0)\\&\times[-i(\hat{p}-\hat{k})+m]\delta[(p-k)^2+m^2][-i\hat{p}+m]\end{aligned}$$

which indeed equals the result for the cut diagram. The minus sign for the closed fermion loop appears here as a minus sign in front of the antiparticle spinor summation.

9. SOME GEOMETRICAL AND PHYSICAL CONSEQUENCES OF SPACE-TIME NONCOMMUTATIVITY

9.1. Specific Rule of Differentiation and Integration of Noncommutative Functions

9.1.1. Differentiation

Because of noncommutativity of spacetime points a rule of differentiation of noncommutative functions with respect to noncommuting variables is changed as follows.

1. Let $\varphi(x)$ be any smooth function, then its star product in noncommutative spacetime reads

$$\varphi_\theta(x) = \exp\left(\frac{1}{2}\,\ln\,\varphi(x)\right)(\star)_c \exp\left(\frac{1}{2}\,\ln\,\varphi(x)\right) \tag{9.1}$$

Making use of the covariant star product formula (2.12), one gets

$$\varphi_\theta(x) = \varphi(x)\left(1 + \frac{\theta^2}{2}T_1(\varphi) + \frac{\theta^4}{4!}T_2(\varphi) + \cdots\right) \tag{9.2}$$

where

$$\begin{aligned} T_1(\varphi) &= \frac{1}{4}\left\{\varphi^{-2}(x)\left[\partial^2_{\nu\mu}\varphi\cdot\partial^2_{\nu\mu}\varphi - (\Box\varphi)^2\right]\right.\\ &\quad \left. + \varphi^{-3}(x)\left[\Box\varphi\cdot\partial^\mu\varphi\cdot\partial_\mu\varphi - \partial_\nu\varphi\left(\partial_\rho\varphi\partial^2_{\rho\nu}\varphi\right)\right]\right\}, \qquad (9.3)\\ T_2(\varphi) &= \frac{1}{\varphi(x)}\left[\left(\partial^\alpha_x\partial^y_\alpha\right)^2 - \Box_x\Box_y\right]^2\sqrt{\varphi(x)}\sqrt{\varphi(y)}\Big|_{y=x} \end{aligned}$$

Equation (9.2) with (9.3) is basis of differential and integral calculuses in noncommutative spacetime.

2. By definition, differentation of noncommuting functions with respect to noncommuting variables is given by a chain rule:

$$\begin{aligned} \frac{\partial}{\partial x_\mu}\varphi(x) &\Rightarrow \frac{\partial}{\partial x_\mu}\star\varphi(x) = \frac{\partial}{\partial x_\mu}\left\{e^{\frac{1}{2}\,\ln\,\varphi(x)}(\star)c^{e^{\frac{1}{2}\,\ln\,\varphi(x)}}\right\},\\ \frac{\partial^2}{\partial x_\nu\partial x_\mu}\varphi(x) &\Rightarrow \frac{\partial}{\partial x_\nu}\star\frac{\partial}{\partial x_\mu}\star\varphi(x) = \frac{\partial}{\partial x_\nu}\left\{e^{\frac{1}{2}\,\ln\,F_\mu(x)}(\star)c^{e^{\frac{1}{2}\,\ln\,F_\mu(x)}}\right\} \qquad (9.4) \end{aligned}$$

and so on. Here

$$\begin{aligned} F_\mu(x) &= \frac{\partial}{\partial x_\mu}\left\{e^{\frac{1}{2}\,\ln\,\varphi(x)}(\star)_c e^{\frac{1}{2}\,\ln\,\varphi(x)}\right\}\\ &= \frac{\partial}{\partial x_\mu}\left\{1 + \frac{\theta^2}{2!}T_1(\varphi) + \frac{\theta^2}{4!}T_2(\varphi) + \cdots\right\}\varphi(x) \qquad (9.5) \end{aligned}$$

In Eq. (9.5) the function $T_1(\varphi)$ is given by expression (9.3). Similar but complicated formula holds for $T_2(\varphi)$:

$$\begin{aligned} T_2(\varphi) &= \varphi^{-1}(x)\left[\left(\partial^x_\alpha\cdot\partial^\alpha_y\right)^2 - \Box_x\cdot\Box_y\right]^2\\ &\quad\times\left[\exp\left(\frac{1}{2}\,\ln\,\varphi(x)\right)(\star)_c \exp\left(\frac{1}{2}\,\ln\,\varphi(y)\right)\right]\Big|_{y=x} \end{aligned}$$

and etc. One can also write formulas like (9.4) for any order of differential forms:

$$d^n \star f(x) \star g(x) = \underbrace{d \star \ldots \star d}_{n-1\text{terms}} \star d \star f(x) \star g(x) =$$

$$\underbrace{d \star \ldots \star d}_{n-1\text{terms}} \star d \cdot \{\Lambda_{xy} f(x)g(y)|_{y=x}\} = \underbrace{d \star \ldots \star d}_{n-2\text{terms}} \cdot F_x = \ldots$$

where

$$F_x = \left\{ e^{\frac{1}{2} \ln f_1(x)} (\star)_c \, e^{\frac{1}{2} \ln f_1(x)} \right\},$$

$$f_1(x) = d.[\Lambda_{xy} f(x)g(y)|_{y=x}] = \cosh\theta \left(\sqrt{\left(\partial_x^\nu \cdot \partial_\nu^y\right)^2 - \Box_x \Box_y} \right) d \times [f(x)g(y)|_{y=x}]$$

9.1.2. Integration

Integration rule for noncommuting functions over noncommuting variables defines by similar way:

$$I_1 = \int d^4x \star \varphi(x) = \int d^4x \exp\left(\frac{1}{2} \ln \varphi(x)\right) (\star)_c \exp\left(\frac{1}{2} \ln \varphi(x)\right)$$

$$I_2 = \int\int d^4x_1 \star d^4x_2 \star \varphi(x_1, x_2) = \int\int d^4x_1 \, d^4x_2 \, F(x_1, x_2) \tag{9.6}$$

where

$$F(x_1, x_2) = \exp\left(\frac{1}{2} \ln f(x_1, x_2)\right) (\star)_c \exp\left(\frac{1}{2} \ln f(x_1, x_2)\right)$$

and

$$f(x_1, x_2) = \Lambda_{x_2 y} \varphi^{\frac{1}{2}}(x_1, x_2) \varphi^{\frac{1}{2}}(x_1, y)|_{y=x_2}$$

Similar expressions of (9.6) hold for any order of integrals for noncommuting functions over many noncommuting variables $x_1, \ldots, x_n$.

As seen above the star product meaning noncommutative properties of space-time coordinates is cancelled by the covariant star product giving strong correlating variables instead of noncommuting ones.

1. As the next step, we consider some concrete consequences arising from the definitions (9.1) and (9.2). For example, owing to Eqs. (9.1) and (9.2)

the Euclidean distance in three-dimensional space acquires the form

$$\mathrm{X}^2 \Rightarrow \mathrm{X}_\theta^2 = \sqrt{\mathrm{X}^2}(\star)_c\sqrt{\mathrm{X}^2} = \mathrm{X}^2\left(1 - \frac{\theta^2}{\mathrm{X}^4} + \frac{\theta^4}{(x^2)^8} + \cdots\right)$$
$$= \mathrm{X}^2 \cdot \frac{1}{1 + \theta^2/\mathrm{X}^4} \tag{9.7}$$

where we have used the commutation relations:

$$[\hat{x}_i, \hat{x}_j] = \theta_{\tau_{ij,}} \qquad \tau_{ij} = \sigma_i\sigma_j - \sigma_j\sigma_i$$

σ_i is the Pauli matrices.

We point out the geometrical rich character of (9.7) due to the noncommutativity of space. Indeed, single sphere of radius $\mathrm{X}^2 = l^2$ in usual space is decomposed many spheres with different radii in accordance with Eq. (9.7). It means that from the point of view of dimensionality the noncommutative space is equivalent to joint spaces with different dimensionalities. We shall solve Eq. (9.7)

$$\frac{\mathrm{X}^2}{1 + \frac{\theta^2}{\mathrm{X}^4}} = l^2 \tag{9.8}$$

or

$$\lambda^3 - l^2\lambda^2 - \theta^2 l^2 = 0, \qquad x^2 = \lambda \tag{9.9}$$

This cubic equation has three real solutions and therefore we have following set spheres arising from space noncommutativity initially:

$$(r^2)_{11} = \left(\frac{5}{9} - \frac{\theta^2}{l^4}\right) l^2, \qquad (r^2)_{12} = \left(1 + \frac{\theta^2}{l^4}\right) l^2,$$
$$(r^2)_{21} = \left(\frac{20}{27} + \frac{3}{4}\frac{\theta^2}{l^4}\right) l^2, \qquad (r^2)_{22} = \left(\frac{\theta}{il^2} - \frac{1}{2}\frac{\theta^2}{l^4}\right) l^2$$
$$(r^2)_{31} = \left(\frac{20}{27} + \frac{3}{4}\frac{\theta^2}{l^4}\right) l^2, \qquad (r^2)_{32} = \left(\frac{\theta}{il^2} - \frac{1}{2}\frac{\theta^2}{l^4}\right) l^2 \tag{9.10}$$

and there also exist four pseudospheres:

$$(r^2)_{21} = -\left(\frac{8}{27} - \frac{1}{4}\frac{\theta^2}{l^4}\right) l^2, \qquad (r^2)_{22} = -\left(\frac{\theta}{il^2} - \frac{1}{2}\frac{\theta^2}{l^4}\right) l^2$$
$$(r^2)_{31} = -\left(\frac{8}{27} - \frac{1}{4}\frac{\theta^2}{l^4}\right) l^2 \qquad (r^2)_{32} = -\left(\frac{\theta}{il^2} - \frac{1}{2}\frac{\theta^2}{l^4}\right) l^2 \tag{9.11}$$

2. It will be expected that the Coulomb potential is also changed in the noncommutative space

$$\varphi_c(x) \Rightarrow \varphi_c^\theta(r) = \frac{e}{4\pi}\left(\frac{1}{r^{1/2}}(\star)_c\frac{1}{r^{1/2}}\right) \tag{9.12}$$

After some simple calculations by using definitions (9.1) and (9.2), one gets

$$\varphi_c^\theta(r) = \frac{e}{4\pi}\frac{1}{r}\left[1 + \frac{1}{2}\frac{\theta^2}{r^4} + \frac{75}{8}\frac{\theta^4}{r^8} + \cdots\right] \tag{9.13}$$

On the other hand, Eq (9.13) may be understood as a sum of potentials arising from different dimensionality of space:

$$\varphi_c^\theta(r) = \varphi_c^{(3)}(r_3) + \varphi_c^{(7)}(r_7) + \varphi_c^{(11)}(r_{11}) \tag{9.14}$$

The first term $\varphi_c^{(3)}(r_3)$ is the usual Coulomb potential in the three-dimensional space, while other two terms $\varphi_c^{(7)}(r_7)$ and $\varphi_c^{(11)}(r_{11})$ are responsible from 7- and 11-dimensional spaces due to the noncommutative space. It is obvious that our scheme, i.e., decomposition in Eq. (9.14), is invariant with respect to $O(3)$, $O(7)$, and $O(11)$ groups and therefore distances r_3, r_7, and r_{11} can be formally understood as

$$r_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad r_7 = \sqrt{x_1^2 + \cdots + x_7^2}, \quad r_{11} = \sqrt{x_1^2 + \cdots + x_{11}^2}$$

It can be seen easily that

$$\text{divgrad}\,\varphi_c^\theta(r) = \text{divgrad}\,\left(\varphi_c^{(7)}(r_7) + \varphi_c^{(11)}(r_{11})\right) \neq 0 \tag{9.15}$$

as it will be expected in the usual sense, however if we define formal divergence denoted trough Div which acts differently on the potential (9.14) depending on those dimensionality of space, for example,

$$\begin{aligned}\text{Divgrad}\,\varphi_c^\theta(r) = \text{divgrad}\,\varphi_c^{(3)}(r_3) + \text{divgrad}\,\varphi_c^{(7)}(r_7)\\ +\text{divgrad}\,\varphi_c^{(11)}(r_{11}) \equiv 0\end{aligned} \tag{9.16}$$

Since

$$\mathbf{a}^3 = \text{grad}\,\varphi_c^{(3)}(r_3) = \left\{-\frac{ex_1}{4\pi r_3^3}, -\frac{ex_2}{4\pi r_3^3}, -\frac{ex_3}{4\pi r_3^3}\right\},$$

$$\mathbf{a}^7 = \text{grad}\,\varphi_c^{(7)}(r_7) = \left\{-\frac{5\theta^2}{8\pi}\,e\frac{x_1}{r^7}, -\frac{5\theta^2}{8\pi}\,e\frac{x_2}{r^7}, \cdots, -\frac{5\theta^2}{8\pi}\,e\frac{x_7}{r^7}\right\},$$

$$\mathbf{a}^{11} = \operatorname{grad}\varphi_c^{(11)}(r_{11}) = \left\{-\frac{9.75}{32\pi}\theta^4\frac{ex_1}{r^{11}}, -\frac{9.75}{32\pi}\theta^4\frac{ex_2}{r^{11}}, \cdots, -\frac{9.75}{32\pi}\theta^4\frac{ex_{11}}{r^{11}}\right\} \tag{9.17}$$

From that it follows

$$\operatorname{div}\mathbf{a}^{(3)} = -\frac{e}{4\pi r^5}\left(3r_3^2 - 3x_1^2 - 3x_2^2 - 3x_3^2\right) = 0,$$

$$\operatorname{div}\mathbf{a}^{(11)} = -\frac{5\theta^2}{8\pi r^9}\left(7r_7^2 - 7x_1^2 - 7x_2^2 - \ldots - 7x_7^2\right) = 0,$$

$$\operatorname{div}\mathbf{a}^{(11)} = -\frac{9.75}{32\pi r^{13}}\left(11r_{11}^2 - 11x_1^2 - 11x_2^2 - \ldots - 11x_{11}^2\right) = 0 \tag{9.18}$$

Above statements are valid due to the usual definition of grad, div, etc., in the c-number space. However, vector and tensor calculuses are also changed in the noncommutative space. We now turn to this problem.

9.2. Vector and Geometrical Meaning of the ⋆-Product

Let us consider three-dimensional noncommutative space in which constant vector **a** defines as

$$a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{K} \tag{9.19}$$

where $a_i(i = x, y, z)$ are constant c-numbers, while unit three vectors $n_i(i = i, j, k)$ obey commutation relations:

$$[n_I, \star n_j] = \theta_{\tau_{ij}} \tag{9.20}$$

Here τ_{ij} and θ are constant antisymmetric three-tensor and dimensionless scale. In our case

$$\tau_{ij} = \sigma_i\sigma_j - \sigma_i\sigma_j, \qquad \sigma_i\sigma_j = 2i\varepsilon_{ijk}\sigma_k \tag{9.21}$$

ε_{ijk} is full antisymmetric unit tensor $\varepsilon_{123} = +1$. In virtue of (9.20), one gets

$$\mathbf{i}\star\mathbf{i} - \mathbf{i}\star\mathbf{i} = 0, \qquad \mathbf{i}\star\mathbf{j} - \mathbf{j}\star\mathbf{i} = \theta_{\tau 12}$$
$$\mathbf{i}\star\mathbf{k} - \mathbf{k}\star\mathbf{i} = \theta_{\tau_{13}}, \qquad \mathbf{k}\star\mathbf{i} - \mathbf{i}\star\mathbf{k} = \theta_{\tau_{31}}$$
$$\mathbf{j}\star\mathbf{k} - \mathbf{k}\star\mathbf{j} = \theta_{\tau_{23}} \tag{9.22}$$

and so on. Here the star product means the star product of the scalar type for vector **n**. Thus, the scalar star product of two constant vectors **a** and **b** is given by

$$\mathbf{a}\star\mathbf{b} = (a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}) \star (b_x \cdot \mathbf{i} + b_y \cdot \mathbf{j} + b_z \cdot \mathbf{k})$$
$$= \mathbf{b}\star\mathbf{a} + 2i\theta[\mathbf{a}\times\mathbf{b}]^i \cdot \sigma_i \tag{9.23}$$

where $[\mathbf{a} \times \mathbf{b}]$ is the usual vector product of $\mathbf{a}$ and $\mathbf{b}$—two vectors. In Eq. (9.23) we have used commutation relations (9.20) and (9.21). Formula (9.23) means that

$$[\mathbf{a}, \star\mathbf{b}]_- = \mathbf{a} \star \mathbf{b} - \mathbf{b} \star \mathbf{a} = 2i\theta[\mathbf{a} \times \mathbf{b}]^i \cdot \sigma_i \tag{9.24}$$

This relation defines geometrical meaning of the scalar star product of two vectors. This commutator equals to zero when two vectors are parallel. There exist also cyclic relations:

$$\begin{aligned} \mathbf{a} \star \mathbf{i} - \mathbf{i} \star \mathbf{a} &= 2i\theta(a_z\sigma_2 - a_y\sigma_3) \\ \mathbf{a} \star \mathbf{j} - \mathbf{j} \star \mathbf{a} &= 2i\theta(a_x\sigma_3 - a_z\sigma_1) \\ \mathbf{a} \star \mathbf{k} - \mathbf{k} \star \mathbf{a} &= 2i\theta(a_y\sigma_1 - a_x\sigma_2) \end{aligned} \tag{9.25}$$

Moreover, if coordinate vectors *x,y,z* satisfying commutation relations

$$\begin{aligned} [xy - yx] &= \theta_{\tau 12}, \\ [xz - zx] &= \theta_{\tau 13}, \\ [yz - zy] &= \theta_{\tau 23} \end{aligned} \tag{9.26}$$

and its the dimensionful parameter θ are dependent on the time variable then other relations are valid

$$\begin{aligned} \dot{x}y - \dot{y}x &= \frac{1}{2}\dot{\theta}\tau_{12}, \qquad x\dot{y} - y\dot{x} = \frac{1}{2}\dot{\theta}\tau_{12} \\ \ddot{x}y - \ddot{y}x &= \frac{1}{4}\ddot{\theta}\tau_{12}, \qquad x\ddot{y} - y\ddot{x} = \frac{1}{4}\ddot{\theta}\tau_{12} \\ &\qquad \dot{x}\dot{y} - \dot{y}\dot{x} = \frac{1}{4}\ddot{\theta}_{\tau_{12}} \end{aligned} \tag{9.27}$$

$$\begin{aligned} \dot{x}z - \dot{z}x &= \frac{1}{2}\dot{\theta}\tau_{13}, \qquad x\dot{z} - z\dot{x} = \frac{1}{2}\dot{\theta}\tau_{13} \\ \ddot{x}z - \ddot{z}x &= \frac{1}{4}\ddot{\theta}\tau_{13}, \qquad x\ddot{z} - z\ddot{x} = \frac{1}{4}\ddot{\theta}\tau_{13} \\ &\qquad \dot{x}\dot{z} - \dot{z}\dot{x} = \frac{1}{4}\ddot{\theta}_{\tau_{13}} \end{aligned} \tag{9.28}$$

$$\begin{aligned} \dot{y}z - \dot{z}y &= \frac{1}{2}\dot{\theta}\tau_{23}, \qquad y\dot{z} - z\dot{y} = \frac{1}{2}\dot{\theta}\tau_{23} \\ \ddot{y}z - \ddot{z}y &= \frac{1}{4}\ddot{\theta}\tau_{23}, \qquad y\ddot{z} - z\ddot{y} = \frac{1}{4}\ddot{\theta}\tau_{23} \\ &\qquad \dot{y}\dot{z} - \dot{z}\dot{y} = \frac{1}{4}\ddot{\theta}_{\tau_{23}} \end{aligned} \tag{9.29}$$

In particular, from relation (9.26) it follows

$$[\mathbf{r}(\star)_\nu \mathbf{r}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x & y & z \end{vmatrix} = \mathbf{i}(yz - zy) - \mathbf{j}(xz - zx)$$

$$+ \mathbf{k}(xy - yx) = \mathbf{i}\theta_{\tau 23} - \mathbf{j}\theta_{\tau 13} + \mathbf{k}\theta_{\tau 12}$$

$$= 2i\theta(\mathbf{i}\sigma\mathbf{1} + \mathbf{j}\sigma\mathbf{2} + \mathbf{k}\sigma\mathbf{3}) = 2i\theta \cdot \vec{\sigma} \tag{9.30}$$

where $\vec{\sigma}$ is the Pauli vector with components σ_1, σ_2, and σ_3.

The vector star product for two vectors **a** and **b** is given by

$$[\mathbf{a}(\star)_\nu \mathbf{b}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ a_x & a_y & a_z \end{vmatrix}_\star = \mathbf{i}(a_y \star b_z - a_z \star b_y)$$

$$-\mathbf{j}(a_x \star b_z - a_z \star b_x) + \mathbf{k}(a_x \star b_y - a_y \star b_x) \tag{9.31}$$

Finally, in accordance with formulas (9.27)–(9.29) we would like to write yet one relation for the radius vector depending on time variable

$$[\dot{\mathbf{r}}(\star)_\nu \dot{\mathbf{r}}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} = \mathbf{i}(\dot{y}\dot{z} - \dot{z}\dot{y}) - \mathbf{j}(\dot{x}\dot{z} - \dot{z}\dot{y})$$

$$+\mathbf{k}(\dot{x}\dot{y} - \dot{y}\dot{x}) = \frac{1}{2}\ddot{\theta}\vec{\sigma} \cdot i \tag{9.32}$$

9.3. Motion of a Material Point in the Noncommutative Space

Let us consider a variable vector $\mathbf{a} = a(x, y, z, t)$ depending on the space coordinates $x_i = (x, y, z)$ and usual time t. In this case, usual vector analysis can be easily generalized by using the covariant $\star$-product. In the noncommutative model the vector velocity and acceleration of the material point have the standard from

$$\mathbf{V}(t) = d\mathbf{r}/dt, \qquad \vec{\omega}(t) = d\mathbf{V}/dt$$

and therefore the Newtonian law has the similar from

$$m d\mathbf{V}/dt = \mathbf{F}$$

or

$$m\vec{\omega} = m\dot{\mathbf{V}} = m\ddot{\mathbf{r}} = \mathbf{F} \tag{9.33}$$

Multiply both parts of Eq. (9.33) by the radius vector $\mathbf{r}$ with the vectorial star product:

$$[\mathbf{r}(\star)_\nu m\ddot{\mathbf{r}}] = [\mathbf{r}(\star)_\nu \mathbf{F}] \tag{9.34}$$

Taking into account Eq. (9.32) with the constant parameter $\theta = \text{const}$ and making use of the identity

$$\begin{aligned} \frac{d}{dt}[\mathbf{r}(\star)_\nu m\dot{\mathbf{r}}] &= [\dot{\mathbf{r}}(\star)_\nu m\dot{\mathbf{r}}] + [\mathbf{r}(\star)_\nu m\ddot{\mathbf{r}}] \\ &= \frac{1}{2}\ddot{\theta} \cdot m\vec{\sigma} + [\mathbf{r}(\star)_\nu m\ddot{\mathbf{r}}] = [\mathbf{r}(\star)_\nu m\ddot{\mathbf{r}}] \end{aligned} \tag{9.35}$$

(since $\ddot{\theta} = 0$) one gets the standard form

$$\frac{d}{dt}[\mathbf{r}(\star)_\nu m\dot{\mathbf{r}}] = [\mathbf{r}(\star)_\nu \mathbf{F}] \tag{9.36}$$

If the force $\mathbf{F}$ belongs along or backward with respect to the direction of the radius vector

$$\mathbf{F} = \gamma \mathbf{r} \tag{9.37}$$

then

$$[\mathbf{r}(\star)_\nu \dot{\mathbf{r}}] = 2i\theta\gamma \frac{\vec{\sigma}}{m} t + C \tag{9.38}$$

where we have used Eq. (9.30). In the usual commutative space Eq. (9.38) with $\theta = 0$ is called the integral of conservation of areas. On the contrary, owing to Eq. (9.38) in the noncommutative model, conservation of areas does not valid. This is one of consequences due to the space noncommutativity in classical physics.

Let us consider yet one consequence for the motion of the material point in the noncommutative space. Multiply the basic Eq. (9.33) by the vector $\mathbf{V}dt = d\mathrm{r}$ with using the scalar star product and obtain

$$m\dot{\mathbf{V}} \star \mathbf{V}dt = \mathbf{F} \star d\mathbf{r}$$

Since $\dot{\mathbf{V}}dt = d \star \mathbf{V}$ and therefore

$$m(\mathbf{V} \star d \star \mathbf{V}) = \mathbf{F} \star d\mathbf{r}$$

By definition

$$2(\mathbf{V} \star d \star \mathbf{V}) = d \star (\mathbf{V} \star V)$$

and

$$d \star (\mathbf{V} \star \mathbf{V}) = d\left[\cosh\theta \left(\sqrt{\left(\nabla_r^i \nabla_i^R\right)^2 - \Delta_r \Delta_R}\right) \cdot (\dot{r}_j \cdot \dot{R}_j)|_{R=r}\right]$$

It can be seen easily that

$$d \star (\mathbf{V} \star \mathbf{V}) = d\frac{V^2}{2} \qquad (9.39)$$

since

$$\frac{\partial}{\partial x_i}\dot{r}_j = \frac{d}{dt}\delta_{ij} = 0$$

On the other hand

$$\mathbf{F} \star \mathbf{r} = \cosh\theta\sqrt{\left(\nabla_r^i \nabla_i^R\right)^2 - \Delta_r \Delta_R} \cdot F^j(r)\, dR_j|_{R=r} = \mathbf{F}\cdot\mathbf{r} + G_\theta$$

where

$$\mathbf{G}_\theta = \left[\cosh\theta\sqrt{\left(\nabla_r^i \nabla_i^R\right)^2 - \Delta_r \Delta_R} - 1\right] dR_j \cdot F^j(r)|_{R=r}$$

Collecting these results, one gets

$$d\left(\frac{mV^2}{2}\right) = (1 - \theta^2 f(r))\mathbf{F}\cdot d\mathbf{r} \qquad (9.40)$$

Here a function $f(r)$ has arisen from the function G_0 in the θ^2-approximation and depends on a concrete form of the force $F(r)$. For example, if the force $\mathbf{F}(r)$ is given Eq. (9.37) then $f(r) = 0$. The expression $\frac{1}{2}mv^2$ is called the living force of the material point and the scalar product $\mathbf{F}\cdot\mathbf{r}$ presents an elementary work of the force $\mathbf{F}$ through displacement $d\mathbf{r}$.

Finally, one can rewrite the Newtonian law in the form

$$dm\mathbf{V} = \mathbf{F}\, dt$$

and integrate its both parts over the limits from the moment t_0 to the moment t. The result reads

$$m\mathbf{V} - m\mathbf{V}_0 = \int_{t_0}^{t} \mathbf{F}\, dt$$

The integral of the force $\mathbf{F}$ over time, i.e., the integral

$$\mathbf{I} = \int_{t_0}^{t} \mathbf{F}\, dt$$

is called momentum of the force $\mathbf{F}$ during the time interval $t - t_0$. In the noncommutative space this is the same as in the usual case.

9.4. Appearance of Inertia as a Residual Effect due to the Noncommutative Space

The origin of inertia presents one of the fundamental problems of physical theory. Newton and Mach considered this problem in different ways. Newton assumed that inertial forces such as centrifugal ones must appear because of acceleration with respect to "absolute space," while Mach suggested that inertial forces are more probably generated by the general mass of heavenly bodies. The difference in their assertions is not metaphysical but physical, since if Mach were right then a large mass would give rise to small alterations of the inertial forces near it, while if Newton were right, then such effects would not appear (for details and further discussion, see Bertotti *et al.*, 1984, Weinberg, 1972.

Here our gaol is to show that the origin of the inertial force may be understood as a residual (or averaging) effect due to the noncommutative space at large distances. Without loss of generality, we suppose that space noncommutativity is based on the following relations

$$[\hat{x}_i, \hat{x}_j] = G_{\mathrm{N}\tau ij} \tag{9.41}$$

where G_N is the Newtonian constant and τ_{ij} is given by Eq. (9.21).

Second assumption is that motion of any bodies in such space is considered as a motion in a continuous medium like liquid and those velocity depends on coordinate variables x_i and t:

$$\mathbf{v}(t) \Rightarrow \mathbf{v}(x_i, t)$$

and therefore velocity of bodies becomes noncommutative variables and we shall understand the Newtonian Eq. (9.33) as an equation with the star product

$$m\,\frac{d}{dt}\mathbf{v}^\star(x_i, t) = \mathbf{F} \tag{9.42}$$

or in components:

$$\begin{aligned} m\,\frac{d}{dt}\nu^\star_x(x_i, t) &= F_x \\ m\,\frac{d}{dt}\nu^\star_y(x_i, t) &= F_y \\ m\,\frac{d}{dt}\nu^\star_z(x_i, t) &= F_z \end{aligned} \tag{9.43}$$

As before, in accordance with Eq. (9.1) one defines

$$v^\star_j(x_i, t) \Rightarrow v^G_j(x_i, t) = \exp\left(\frac{1}{2}\,\ln v_j(x_i, t)\right)(\star)_c \exp\left(\frac{1}{2}\,\ln v_j(x_i, t)\right) \tag{9.44}$$

$(j = x, y, z)$. Here in our particular case, the parameter θ is equal to the Newtonian constant G_N.

Thus, space noncommutativity gives rise nonlinear self-turbulence version of the Newtonian equation with some internal force arisen from Eqs. (9.2) and (9.3) for spatial components of the velocity:

$$m\,\frac{dv_i}{dt} = F_i^{\rm ext} + F_i^{\rm int} \tag{9.45}$$

where

$$\begin{aligned} F_i^{\rm int} = & -\frac{G_{\rm N}^2}{8} m\,\frac{d}{dt}\left\{v_i^{-1}(X,t)\left[\partial_{kj}^2 v_i\,\partial_{kj}^2 v_i - (\Delta v_i)^2\right]\right.\\ & \left. + v_i^{-2}(X,t)\left[\Delta v_i\cdot\partial^j v_i\,\partial_j v_i - \partial_j vi\left(\partial_k v_i\,\partial_{kj}^2 v_i\right)\right]\right\} \end{aligned} \tag{9.46}$$

Here $i = x, y, z$. We see that the force (9.46) depends nonlinearly on velocity field and is negligible small due to the factor $G_{\rm N}^2$ with respect to measurable effects in classical physics for any values of the velocity except some extremal conditions: at the singular point v = 0 and changing its direction quickly. For example, terms like $-v_i^{-2}(\mathbf{x},t)\dot{v}_i$ and $-2v_i^{-3}(\mathbf{x},t)\dot{v}_i$ in Eq. (9.46) give similar δ-function effect $\delta(v)$ at $v\to 0$. This reflects exactly innermost specific properties of the inertial force. It seems that the origin of inertia is linked with pure spacetime properties, namely its noncommutative nature. This fact is very interesting and more attractive. Even if time is noncommutative, then Eqs. (9.41), (9.42), and (9.45) are valid with a little difference:

$$[\hat{x}_\nu, \hat{x}_\mu] = G_{\rm N}\tau_{\nu\mu}$$

$$m\frac{d}{dt}\star v^\star(x,t) = \mathbf{F} \tag{9.47}$$

$$m\frac{dv_i}{dt} = F_i^{\rm ext} + \tilde{F}_i^{\rm ext} \tag{9.48}$$

where

$$\tau_{\nu\mu} = \gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu$$

and

$$\begin{aligned} \tilde{F}_i^{\rm int} = & -\frac{G_{\rm N}^2}{8} m\,\frac{d}{dt}\left\{v_i^{-1}(X,t)\left[\partial_{\nu\mu}^2 v_i\,\partial_{\nu\mu}^2 v_i - (\Box v_i)^2\right]\right.\\ & \left. + v_i^{-2}(X,t)\left[\Box v_i\cdot\partial^\mu v_i\,\partial_\mu v_i - \partial_\mu vi(\partial_\rho v_i\,\partial_{\rho\nu}^2 v_i)\right]\right\} \end{aligned} \tag{9.49}$$

Hidden forces (9.46) and (9.49) are responsible for inertia but do not detectable in process of motion of bodies except for specific moments: changing in direction or absolute value of those velocity quickly (in particular, from which a centrifugal force is arisen). However, an external force switches off or on at an instant time and at the same time the hidden force does rech its largest value at such short moment, after that it turns to zero quickly.

9.5. Differential Operatiors in the Noncommutative Space

Our next purpose is to find operation of differential operators like grad, div, rot, $\Delta = \nabla^2$ in the noncommutative space, which played an important role in the noncommutative field theory.

9.5.1. Gradient or the Hamiltonian Operator (Nabla Operator)

In the noncommutative space, gradient of a scalar function is defined by using the $\star$-product

$$\mathrm{grad}_\theta \varphi = \mathrm{grad} \star \varphi = \nabla \star \varphi = \mathbf{i}\frac{\partial}{\partial x} \star \varphi + \mathbf{j}\frac{\partial}{\partial y} \star \varphi + \mathbf{k}\frac{\partial}{\partial z} \star \varphi \tag{9.50}$$

where $\varphi = \varphi(x, y, z, t)$ is a scalar field.

Further, it is easily seen that by means of this operator one can express gradient of one vector by an another vector

$$(\mathbf{v}(\star)_s \nabla) \star \mathbf{a} = v_x \star \frac{\partial}{\partial x} \star \mathbf{a} + v_y \star \frac{\partial}{\partial y} \star \mathbf{a} + v_z \star \frac{\partial}{\partial z} \star \mathbf{a} \tag{9.51}$$

The divergence of the vector a may be formally considered as the scalar star product of the symbolic vector ∇ on the vector **a**

$$\mathbf{a} = \mathbf{i}a_x + \mathbf{j}a_y + \mathbf{k}a_z$$

Indeed, carrying out its remultiplication by the formula for the scalar (star) product of two vectors

$$\mathbf{b} \star \mathbf{a} = b_x \star a_x + b_y \star a_y + b_z \star a_z$$

and assuming

$$b_x = \frac{\partial}{\partial x}, \quad b_y = \frac{\partial}{\partial y}, \quad b_z = \frac{\partial}{\partial z}$$

one gets

$$\nabla \star \mathbf{a} = \frac{\partial}{\partial x} \star a_x + \frac{\partial}{\partial y} \star a_y + \frac{\partial}{\partial z} \star a_z = \mathrm{div} \star \mathbf{a} \tag{9.52}$$

Now we would like to change the star product in above formulas by its covariant version $(\star)_c$ and after that *grad*, *div*, etc., are considered as the standard operations.

1. $$\mathrm{grad}_\theta \varphi = \mathrm{grad}\left\{\exp\left(\frac{1}{2}\ln\varphi(x)\right)(\star)_c \exp\left(\frac{1}{2}\ln\varphi(x)\right)\right\}$$

2.
$$\begin{aligned}\{(\mathbf{v}\cdot\nabla)\mathbf{a}\}_x &= v_x \star \frac{\partial}{\partial x}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\}\\ &\quad + v_y \star \frac{\partial}{\partial y}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\}\\ &\quad + v_z \star \frac{\partial}{\partial z}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\}\\ &= v_x\left\{\exp\left(\frac{1}{2}\ln f_x^x\right)(\star)_c \exp\left(\frac{1}{2}\ln f_x^x\right)\right\}\\ &\quad + v_y\left\{\exp\left(\frac{1}{2}\ln f_x^y\right)(\star)_c \exp\left(\frac{1}{2}\ln f_x^y\right)\right\}\\ &\quad + v_z\left\{\exp\left(\frac{1}{2}\ln f_x^z\right)(\star)_c \exp\left(\frac{1}{2}\ln f_x^z\right)\right\} \end{aligned} \tag{9.53}$$

and similar formulas hold for y- and z-components. Here

$$\begin{aligned} f_x^x &= \frac{\partial}{\partial x}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\},\\ f_x^y &= \frac{\partial}{\partial y}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\},\\ f_x^z &= \frac{\partial}{\partial z}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\} \end{aligned}$$

While Eq. (9.52) takes the form

$$\begin{aligned}\nabla\star\mathbf{a} \Rightarrow &\frac{\partial}{\partial x}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_x\right)\right\}\\ &+\frac{\partial}{\partial y}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_y\right)\right\}\\ &+\frac{\partial}{\partial z}\left\{\exp\left(\frac{1}{2}\ln a_x\right)(\star)_c \exp\left(\frac{1}{2}\ln a_z\right)\right\} \end{aligned} \tag{9.54}$$

Consider some examples. Let $\varphi(x, y, z)$ be the length of radius vector in the three-dimensional space:

$$\varphi = \sqrt{x^2 + y^2 + z^2} = r$$

Then, by definition

$$\mathbf{a}_\theta = \nabla\star r \Rightarrow \nabla\left\{\exp\left(\frac{1}{2}\ln r\right)(\star)_c \exp\left(\frac{1}{2}\ln r\right)\right\} \tag{9.55}$$

Direct calculations give

$$r_\theta = \exp\left(\frac{1}{2}\ln r\right)(\star)_c \exp\left(\frac{1}{2}\ln r\right)$$
$$= r\left(1 + \frac{3}{8}\frac{\theta^2}{r^4} + \frac{2377}{8.64}\frac{\theta^4}{r^8} + \cdots\right) \tag{9.56}$$

and therefore

$$\mathbf{a}_\theta = \frac{\mathbf{r}}{r}\left[1 - \frac{9\theta^2}{8\pi^4} - \frac{2377 \cdot 7}{8 \cdot 64}\frac{\theta^4}{r^8}\right] \tag{9.57}$$

This is an explicit form of the gradient of the Euclidean distance r in the noncommutative space.

The second example is the gradient of the modified Coulomb law (9.12) or (9.13) in the noncommutative model:

$$\mathbf{E}_\theta = -\nabla \star \frac{e}{4\pi}\frac{1}{r} = \frac{e}{4\pi r^3}\mathbf{r}\left[1 + \frac{5}{2}\frac{\theta^2}{r^4} + \frac{75 \cdot 9}{8}\frac{\theta^4}{r^8}\right] \tag{9.58}$$

This is an electric static field of the point-like charge e in the noncommutative space.

9.5.2. Whirl (or Rotor) of a Vector and the Laplacian Operator in the Noncommutative Space

Rotor of a vector in the noncommutative model is defined by using the vectorial star product:

$$\text{rot}_\theta \mathbf{a} = \nabla(\star)_v \mathbf{a} = \mathbf{i}\left(\frac{\partial}{\partial y} \star a_z - \frac{\partial}{\partial z} \star a_y\right)$$
$$+\mathbf{j}\left(\frac{\partial}{\partial z} \star a_x - \frac{\partial}{\partial x} \star a_z\right) + \mathbf{k}\left(\frac{\partial}{\partial x} \star a_y - \frac{\partial}{\partial y} \star a_x\right) \tag{9.59}$$

We know that from the pure geometrical point of view this differential operator in the usual commutative space possesses remarkable property:

$$\text{rotgrad}\varphi = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi = 0 \tag{9.60}$$

for any scalar field. It is easy to see that this fundamental property of space does not valid in the noncommutative space.

Let us consider two concrete examples:

$$\mathbf{a}^{(1)} = \nabla \star r = \frac{\mathbf{r}}{r}\left[1 - \frac{9}{8}\frac{\theta^2}{r^4} - \frac{2377.7}{8.64}\frac{\theta^4}{r^8}\right] \tag{9.61}$$

and

$$\mathbf{a}^{(3)} = -\frac{e}{4\pi}\nabla \star \frac{1}{r} = \frac{e}{4\pi r^3}\mathbf{r}\left[1 + \frac{5}{2}\frac{\theta^2}{r^4} + \frac{75.9}{8.}\frac{\theta^4}{r^8}\right] \tag{9.62}$$

We would like to calculate rotor of these vectors by the formula

$$\begin{aligned}\text{rot} \star \mathbf{a}^{(i)} = \mathbf{i}\big(\partial_y \star a_z^{(i)} - \partial_z \star a_y^{(i)}\big) + \mathbf{j}\big(\partial_z \star a_x^{(i)} - \partial_x \star a_z^{(i)}\big)\\ + \mathbf{k}\big(\partial_x \star a_y^{(i)} - \partial_y \star a_x^{(i)}\big)\end{aligned} \tag{9.63}$$

where by definition of the covariant star product

$$\partial_y \star \mathbf{a}_z^{(i)} = \partial_y \left\{\exp\left(\frac{1}{2}\,\ln a_z^{(i)}\right)(\star)_c \exp\left(\frac{1}{2}\,\ln a_z^{(i)}\right)\right\} \tag{9.64}$$

and so on. First of all, we calculate the following expressions:

$$\begin{aligned}A_z^{(i)} &= \exp\left(\frac{1}{2}\,\ln a_z^{(i)}\right)(\star)_c \exp\left(\frac{1}{2}\,\ln a_z^{(i)}\right)\\ &= z\lambda^{-i/2-3}(y^2 + x^2)N(i) + \frac{i}{16}\frac{1}{z}(x^2 + y^2)\lambda^{-i/2-2}\\ &\quad + \frac{i}{4z}\lambda^{-i/2-1}\left(-\frac{3}{4} + \frac{1}{4}i\right) + \frac{i^2}{8}z\lambda^{-i/2-2}\\ &\quad \times\left[-\frac{1}{2} - i - \frac{4}{i} - 2\left(\frac{i}{4} - 3\right)^2\right]\end{aligned} \tag{9.65}$$

$$\begin{aligned}A_x^{(i)} &= x\lambda^{-i/2-3}(y^2 + z^2)N(i) + \frac{i}{16}\frac{1}{x}(z^2 + y^2)\lambda^{-i/2-2}\\ &\quad + \frac{i}{4x}\lambda^{-i/2-1}\left(-\frac{3}{4} + \frac{1}{4}i\right) + \frac{i^2}{8}x\lambda^{-i/2-2}\\ &\quad \times\left[-\frac{1}{2} - i - \frac{4}{i} - 2\left(\frac{i}{4} - 3\right)^2\right]\end{aligned} \tag{9.66}$$

$$\begin{aligned}A_y^{(i)} &= y\lambda^{-i/2-3}(z^2 + x^2)N(i) + \frac{i}{16}\frac{1}{y}(x^2 + z^2)\lambda^{-i/2-2}\\ &\quad + \frac{i}{4y}\lambda^{-i/2-1}\left(-\frac{3}{4} + \frac{1}{4}i\right) + \frac{i^2}{8}y\lambda^{-i/2-2}\\ &\quad \times\left[-\frac{1}{2} - i - \frac{4}{i} - 2\left(\frac{i}{4} - 3\right)^2\right]\end{aligned} \tag{9.67}$$

From these equations it follows $A_x^{(i)} = A_z^{(i)}(z \to x, x \to z)$ and $A_y^{(i)} = A_z^{(i)}(z \to y, y \to z)$. Here $\lambda = x^2 + y^2 + z^2$, $N(i) = \frac{1}{4}i^2(i-1)(\frac{1}{4}i +$

$1) - \frac{1}{4}i^2$, $i = 1$, and $i = 3$ for (9.61), and (9.62), respectively. Further, taking into account formulas (9.64), (9.65)–(9.67), one can calculate rotor of vectors (9.61) and (9.62) by using Eq. (9.63)

$$\begin{aligned}
&\text{rot}_\theta \mathbf{a}^{(1)} = \text{rot} \star \mathbf{a}^{(1)} = \text{rot} \star \text{grad} \star r = \frac{\theta^2}{2} \\
&\times \left\{ \mathbf{i} \left[\frac{7}{16} zy(z^2 - y^2) r^{-9} + \frac{1}{2} \left(\frac{y}{z} - \frac{z}{y} \right) r^{-5} - \frac{5}{16} \left(\frac{y}{z}(x^2 + y^2) - \frac{z}{y}(x^2 + z^2) \right) r^{-7} \right] \right. \\
&+ \mathbf{j} \left[\frac{7}{16} xz(x^2 - z^2) r^{-9} + \frac{1}{2} \left(\frac{z}{x} - \frac{x}{z} \right) r^{-5} - \frac{5}{16} \left(\frac{z}{x}(y^2 + z^2) - \frac{x}{z}(y^2 + x^2) \right) r^{-7} \right] \\
&\left. + \mathbf{k} \left[\frac{7}{16} yx(y^2 - x^2) r^{-9} + \frac{1}{2} \left(\frac{x}{y} - \frac{y}{x} \right) r^{-5} - \frac{5}{16} \left(\frac{x}{y}(z^2 + x^2) - \frac{y}{z}(z^2 + y^2) \right) r^{-7} \right] \right\}
\end{aligned} \tag{9.68}$$

and

$$\begin{aligned}
\text{rot}_\theta \mathbf{a}^{(3)} &= \text{rot} \star \mathbf{a}^{(3)} = \text{rot} \star \text{grad} \star \left(-\frac{e}{4\pi} \frac{1}{r} \right) \\
&= \frac{e}{4\pi} \frac{\theta^2}{2} \left\{ \mathbf{i} \left[\frac{243}{16} zy(z^2 - y^2) r^{-11} + \frac{3}{8} \left(\frac{y}{z} - \frac{z}{y} \right) r^{-7} \right. \right. \\
&\quad \left. - \frac{21}{16} \left(\frac{y}{z}(x^2 + y^2) - \frac{z}{y}(x^2 + z^2) \right) r^{-9} \right] \\
&\quad + \mathbf{j} \left[\frac{243}{16} xz(x^2 - z^2) r^{-11} + \frac{3}{8} \left(\frac{z}{x} - \frac{x}{z} \right) r^{-7} \right. \\
&\quad \left. - \frac{21}{16} \left(\frac{z}{x}(y^2 + z^2) - \frac{x}{z}(y^2 + x^2) \right) r^{-9} \right] \\
&\quad + \mathbf{k} \left[\frac{243}{16} yx(y^2 - x^2) r^{-11} + \frac{3}{8} \left(\frac{x}{y} - \frac{y}{z} \right) r^{-7} \right. \\
&\quad \left. \left. - \frac{21}{16} \left(\frac{x}{y}(z^2 + x^2) - \frac{y}{x}(z^2 + y^2) \right) r^{-9} \right] \right\}
\end{aligned} \tag{9.69}$$

From Eqs. (9.68) and (9.69) it is immediately seen that rotgrad does not identitically zero in the noncommutative space.

For completeness, we want to calculate div$\star$ grad $\star\varphi$ in the noncommutative model. For two concrete chosen vectors (9.61) and (9.62) it takes the form:

$$\text{div} \star \mathbf{a}^{(3)} = \text{div} \star \text{grad} \star \left(-\frac{e}{4\pi} \frac{1}{r} \right) = -\frac{e}{4\pi} \frac{\theta^2}{2} \left\{ \frac{431}{4} r^{-7} \right.$$

$$-\frac{27}{4}(z^2y^2+z^2x^2+x^2y^2)r^{-11}-\frac{3}{16}\left[\frac{x^2+y^2}{z^2}+\frac{x^2+z^2}{y^2}+\frac{y^2+z^2}{x^2}\right]r^{-7}$$
$$\left.-\frac{81.7}{16}(z^4+y^4+x^4)r^{-11}\right\} \tag{9.70}$$

$$\mathrm{div}\star\mathbf{a}^{(1)}=\mathrm{div}\star\mathrm{grad}\star r=\frac{2}{r}+\frac{\theta^2}{2}\left\{\frac{229}{32}r^{-5}\right.$$
$$+\frac{7}{2}(x^2z^2+z^2y^2+x^2y^2)r^{-9}-\frac{1}{16}\left[\frac{x^2+y^2}{z^2}+\frac{x^2+z^2}{y^2}+\frac{y^2+z^2}{x^2}\right]r^{-5}$$
$$\left.+\frac{1}{8}\left(\frac{1}{z^2}+\frac{1}{y^2}+\frac{1}{x^2}\right)r^{-3}+\frac{35}{36}(z^4+y^4+x^4)r^{-9}\right\} \tag{9.71}$$

respectively. We know that in the usual commutative space *divgrad* $\left(-\frac{e}{4\pi}\frac{1}{r}\right)=0$ while as seen from Eq. (9.70) this identity does not valid for the noncommutative theory.

By definition, we call div $\star$ grad the Laplacian operator in the noncommutative space and denote it through $\Delta_\star$,

$$\Delta_\star=\mathrm{div}\star\mathrm{grad} \tag{9.72}$$

The Laplacian equation

$$\Delta\varphi=0$$

in the noncommutative space satisfies up to order of θ^2:

$$\Delta_\star\varphi=O(\theta^2)$$

or

$$(\nabla(\star)_c\nabla)\varphi=\Delta_\star\varphi=O(\theta^2)$$

where

$$\Delta_\star\varphi=\nabla(\star)_c\nabla\star\varphi=\Delta_\star\star\varphi.$$

This is definition of the Laplacian operator in the noncommutative space.

9.5.3. The Differential Operator divrota and Possible Evidence of the Dirac Monopole in the noncommutative Space

We recall that definition of divergence gives rise identity

$$\mathrm{divrot}\,\mathbf{a}=0$$

in the commutative space. It means that rotor of the vector field for any vector a is free of sources. Therefore basic parts of the Maxwell equation are

$$\mathrm{div}\ \mathbf{E} = 4\pi\rho \quad \text{and} \quad \mathrm{div}\mathbf{H} = 0,$$

where ρ is the density of electric charges.

By analogy with the electric charge we suppose the existence of a magnetic charge g and its magnetic field generated by this charge. The noncommutative space allows us to appearance of such magnetic field given by the vector potential

$$\mathbf{A}_g = -\frac{g}{4\pi}\mathrm{grad}\left(\frac{1}{r}\right) \tag{9.73}$$

Indeed,

$$\mathbf{H}_g = \mathrm{rot}\star\mathbf{A}_g = -\frac{g}{4\pi}\mathrm{rot}\star\mathrm{grad}\star\frac{1}{r} \neq 0 \tag{9.74}$$

However, if we would like to take *div* of the vector (9.69) in the usual sense (without the $\star$-product) then we observe that

$$\begin{aligned}\mathrm{divrot}_\theta\mathbf{a}_g^{(3)} &= \mathrm{div}\mathbf{H}_g = \mathrm{divrot}\star\mathrm{grad}\left(-\frac{g}{4\pi}\frac{1}{r}\right)\\ &= \frac{\partial}{\partial x}\left(\partial_y\star\mathbf{a}_z^{(3)} - \partial_z\star\mathbf{a}_y^{(3)}\right) + \frac{\partial}{\partial y}\left(\partial_z\star\mathbf{a}_x^{(3)} - \partial_x\star\mathbf{a}_z^{(3)}\right)\\ &\quad + \frac{\partial}{\partial z}\left(\partial_x\star\mathbf{a}_y^{(3)} - \partial_y\star\mathbf{a}_x^{(3)}\right)\end{aligned} \tag{9.75}$$

$$\begin{aligned}&= \frac{g}{4\pi}\frac{\theta^2}{2}\left\{-\frac{243.11}{16}xyz(z^2-y^2)r^{-13} - \frac{21}{4}\left(\frac{y}{z}-\frac{z}{y}\right)xr^{-9}\right.\\ &\quad + \frac{189}{16}x\left[\frac{y}{z}(x^2+y^2) - \frac{z}{y}(x^2+z^2)\right]r^{-11}\\ &\quad - \frac{243.11}{16}xyz(x^2-z^2)r^{-13} - \frac{21}{4}\left(\frac{z}{x}-\frac{x}{z}\right)yr^{-9}\\ &\quad + \frac{189}{16}y\left[\frac{z}{x}(y^2+z^2) - \frac{x}{z}(y^2+x^2)\right]r^{-11}\\ &\quad - \frac{243.11}{16}xyz(y^2-x^2)r^{-13} - \frac{21}{4}\left(\frac{x}{y}-\frac{y}{x}\right)zr^{-9}\\ &\quad \left. + \frac{189}{16}z\left[\frac{x}{y}(z^2+x^2) - \frac{y}{x}(z^2+y^2)\right]r^{-11}\right\} \equiv 0\end{aligned} \tag{9.76}$$

This identity means that in the noncommutative space equation

$$\mathrm{div}\star\mathrm{rot}_\theta\mathbf{a}_g^3 = \mathrm{div}\star\mathbf{H}_g = \theta^4\cdot f(x,y,z) \tag{9.77}$$

is almost zero and proportional to the θ^4-order of approximation, where

$$f(x,y,z) = -\frac{g}{4\pi}\left(\frac{\partial H^{(x)}}{\partial x} + \frac{\partial H^{(y)}}{\partial y} + \frac{\partial H^{(z)}}{\partial z}\right) \tag{9.78}$$

and

$$\begin{aligned} H^{(i)} = {} & \left(T'^{(i)}_{yx}\right)^2 + \left(T'^{(i)}_{yz}\right)^2 + \left(T'^{(i)}_{zx}\right)^2 \\ & - T'^{(i)}_{xx}T'^{(i)}_{yy} - T'^{(i)}_{zz}T'^{(i)}_{xx} - T'^{(i)}_{zz}T'^{(i)}_{yy} \end{aligned} \tag{9.79}$$

$(i = x, y, z)$. Let $i = z = 3$ then

$$T^{(3)}_x = -\frac{11}{2}Q_3^{1/2}x\lambda^{-15/4} + \frac{1}{2}Q_3^{-1/2}M^{(3)}_x\lambda^{-11/4},$$

$$\begin{aligned} M^{(3)}_x = {} & -\frac{248}{8}(3x^2y - y^3) + \frac{3}{8}\left(\frac{1}{y} + \frac{y}{x^2}\right)\lambda^2 + \frac{3}{2}\left(\frac{x}{y} - \frac{y}{x}\right)x\lambda \\ & -\frac{21}{16}\left[\frac{z^2+3x^2}{y} + \frac{y}{x^2}(z^2+Y^2)\right]\lambda - \frac{21}{8}\left[\frac{x}{y}(z^2+x^2) - \frac{y}{x}(z^2+y^2)\right]x, \end{aligned}$$

$$T^{(3)}_y = -\frac{11}{2}Q_3^{1/2}y\lambda^{-15/4} + \frac{1}{2}Q_3^{-1/2}M^{(3)}_y\lambda^{-11/4},$$

$$\begin{aligned} M^{(3)}_y = {} & -\frac{243}{8}(3y^2x - x^3) + \frac{3}{8}\left(\frac{x}{y^2} + \frac{1}{x}\right)\lambda^2 + \frac{3}{2}\left(\frac{x}{y} - \frac{y}{x}\right)y\cdot\lambda \\ & -\frac{21}{16}\left[-\frac{x}{y^2}(z^2+x^2) - \frac{1}{x}(z^2+3y^2)\right]\lambda - \frac{21}{8}\left[-\frac{x}{y}(z^2+x^2) - \frac{y}{x}(z^2+y^2)\right]y, \end{aligned}$$

and

$$T^{(3)}_z = -\frac{11}{2}Q_3^{1/2}z\lambda^{-15/4} + \frac{1}{2}Q_3^{-1/2}M^{(3)}_z\lambda^{-11/4},$$

$$\begin{aligned} M^{(3)}_z = {} & \frac{3}{2}\left(\frac{x}{y} - \frac{y}{x}\right)z\cdot\lambda - \frac{21}{8}\left(\frac{xz}{y} - \frac{yz}{x}\right)\lambda \\ & -\frac{21}{8}\left[\frac{x}{y}(z^2+x^2) - \frac{y}{x}(z^2+y^2)\right]z \end{aligned}$$

Moreover, the following notation is clear:

$$T'^{(3)}_{xx} = \frac{\partial}{\partial x}T^{(3)}_x, \quad T'^{(3)}_{xy} = \frac{\partial}{\partial y}T^{(3)}_x, \quad T'^{(3)}_{xz} = \frac{\partial}{\partial z}T^{(3)}_x \tag{9.80}$$

and so on. Here

$$\lambda = x^2 + y^2 + z^2, \qquad r = \lambda^{1/2},$$

$$
\begin{aligned}
a_x = a_1 &= \frac{243}{16} zy(z^2 - y^2) r^{-11} + \frac{3}{8}\left(\frac{y}{z} - \frac{z}{y}\right) r^{-7} \\
&\quad - \frac{21}{16}\left[\frac{y}{z}(x^2 + y^2) - \frac{z}{y}(x^2 + z^2)\right] r^{-9}, \\
a_y = a_2 &= \frac{243}{16} xz(x^2 - z^2) r^{-11} + \frac{3}{8}\left(\frac{z}{x} - \frac{x}{z}\right) r^{-7} \\
&\quad - \frac{21}{16}\left[\frac{z}{x}(y^2 + z^2) - \frac{x}{z}(y^2 + x^2)\right] r^{-9}, \\
a_z = a_3 &= \frac{243}{16} xy(y^2 - x^2) r^{-11} + \frac{3}{8}\left(\frac{x}{y} - \frac{y}{x}\right) r^{-7} \\
&\quad - \frac{21}{16}\left[\frac{x}{y}(z^2 + x^2) - \frac{y}{x}(z^2 + y^2)\right] r^{-9}
\end{aligned}
$$

From (9.79) it is easily seen that

$$H^{(2)} = H^{(3)}(y \to x, x \to z, z \to y)$$

and

$$H^{(1)} = H^{(3)}(x \to y, y \to z, z \to x)$$

It is worth noting that instead of Eq. (9.77) in the noncommutative space the electric field satisfies equation

$$\text{div} \star \text{grad}_\theta \mathbf{a}_e^{(3)} = \text{div} \star \mathbf{E} = 4\pi\rho + O(\theta^2) \tag{9.81}$$

where ρ is the density of the electric charge. Comparing two Eqs. (9.77) and (9.81) one asserts that probability that observation of a magnetic charge if it exists in nature is very small.

9.6. Linear Integral, Flow of a Vector Through Surface, and the Gauss Theorem in the Noncommutative Space

In this section we give some definitions of vector integral calculuses in the noncommutative space, which played an important role in the tensor analysis for the noncommutative theory.

9.6.1. The Linear Integral

The linear integral of the vector **a** along a curve L in the noncommutative space is given by the star product:

$$\int_L d\mathbf{r} \star \mathbf{a} \tag{9.82}$$

where

$$d\mathbf{r} \star \mathbf{a} = dx \star a_x + dy \star a_y + dz \star a_z \tag{9.83}$$

For example, calculate the integral

$$\int_L (dy \star x - dx \star y)$$

taking along the contour of the circle:

$$x^2 + y^2 = R^2$$

Since, by definition:

$$dy \star x = dy \left\{ \exp\left(\frac{1}{2}\ \ln\ x\right)(\star)_c \exp\left(\frac{1}{2}\ \ln\ x\right)\right\} = dy \cdot x$$

and

$$dx \star y = dx \left\{ \exp\left(\frac{1}{2}\ \ln\ y\right)(\star)_c \exp\left(\frac{1}{2}\ \ln\ y\right)\right\} = dx \cdot y,$$

and therefore one can parameterize this integral by using one variable φ:

$$x = R\ \cos\varphi \qquad y = R\ \sin\varphi$$

or

$$x\, dy - y\, dx = R^2 d\varphi$$

The result reads

$$\int_L (dy \star x - dx \star y) = \int_0^{2\pi} R^2 d\varphi = 2\pi R^2.$$

We recall that the linear integral of a vector along a closed curve is called circular of the vector over this curve.

9.6.2. Flow of the Vector Trough a Surface

Flow of the vector a trough the surface S can be written in the form of the star (scalar) product:

$$\int_S d\mathbf{S}(\star)_s \mathbf{a} = \int_S dS \star a_n = \int_S dS\, \mathbf{a}(\star)_s \mathbf{n} \tag{9.84}$$

where **n** is a unit normal vector to the surface S, and

$$a_n = \mathbf{a}(\star)_c \mathbf{n} = a_x \star \cos(\mathbf{n}, x) + a_y \star \cos(\mathbf{n}, y) + a_z \star \cos(\mathbf{n}, z) \tag{9.85}$$

Consequence 1. Let a be the constant vector and if S is the closed surface then

$$\oint_S d\mathbf{S} \star \mathbf{a}_0 = 0 \tag{9.86}$$

Since, one can formally write

$$\mathbf{a}_0 = \exp\left(\frac{1}{2}\ \ln \mathbf{a}_0\right)(\star)_c\ \exp\left(\frac{1}{2}\ \ln \mathbf{a}_0\right) = \mathbf{a}_0 \tag{9.87}$$

and therefore

$$\oint_S \mathbf{a}_0 \cdot d\mathbf{S} = \mathbf{a}_0 \cdot \oint_S d\mathbf{S}$$

As a usual commutative theory, vector over the closed surface is equal to zero, i.e.,

$$\oint_S d\mathbf{S} = 0$$

On the other words,

$$\oint_S \cos(\mathbf{n}, x)\, dS = 0, \quad \oint_S \cos(\mathbf{n}, y)\, dS = 0, \quad \oint_S \cos(\mathbf{n}, z)\, dS = 0$$

Indeed, therefore we obtain Eq. (9.86).

Consequence 2. Let $\mathbf{a} = \mathbf{r}$ be the radius vector of a point. Then

$$\oint_S dS \star \mathbf{r} = 3V + O(\theta^2) \tag{9.88}$$

where V is the volume limited by the closed surface. Since, one can formally write

$$\star\mathbf{r} = \exp\left(\frac{1}{2}\ \ln \mathbf{r}\right)(\star)_c\ \exp\left(\frac{1}{2}\ \ln \mathbf{r}\right) = \mathbf{r}\left(1 + O\left(\frac{\theta^2}{r^4}\right)\right) \tag{9.89}$$

which gives Eq. (9.88).

9.6.3. The Gauss Theorem

In the noncommutative space, the Gauss theorem is generalized in the following star product form:

$$\oint_S dS \star a_n = \oint_S dS[a_x \star \cos(\mathbf{n}, x) + a_y \star \cos(\mathbf{n}, y) + a_z \star \cos(\mathbf{n}, z)]$$
$$= \int_V dV \star \mathrm{div} \star \mathbf{a} \tag{9.90}$$

Finally, it should be noted that because of the noncommutative nature of space, an absolute ideal concept of pure noncompressible liquid does not exist. In this case, the volume of the liquid going out through any surface does not always equal to the volume going in, and therefore full flow is almost zero up to the θ^2-order in the parameter of noncommutativity. Thus instead of the equation of indissolubility of noncompressible liquid

$$\mathrm{div}\mathbf{a} = 0$$

we obtain an approximate equation

$$\mathrm{div} \star \mathbf{a} = \mathrm{div}\mathbf{a} + O(\theta^2) = O(\theta^2)$$

for indissoluble liquid moving in the noncommutative space.

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REFERENCES

Ardalan, E., Arfaei, H., and Shiekh-Jabbari, M. M. (1998). *Preprint* hep-th/9803067.

Ardalan, E., Arfaei, H., and Shiekh-Jabbari, M. M. (1999). *Journal of High Energy Physics* **9902**, 016. *Preprint* hep-th/9810072.

Athanasiu, G. G., Floratos, E. G., and Nicolis, S. (1996). Holomorphic quantization on the torus and finite quantum mechanics. *Journal of Physics A: Mathematical and General* **29**, 6737. *Preprint* hep-th/9509098.

Banerjee, R. (2002). *Modern Physics Letters A* **17**, 631.

Banks, T., Fischler, W., Shenker, S. H., and Susskind, L. (1997). M theory as a matrix model: A conjecture. *Physical Review D: Particles and Fields* **55**, 5112. *Preprint* hep-th/9610043.

Bertotti, B., Farinella, P., Milani, A., Nobili, A. M., and Sacerdote, F. (1984). Linking reference systems from space. *Astron. Astrophy.* **133**, 231–238.

Bigatti, D. and Susskind, L. (2000). Magnetic fields, branes and noncommutative geometry, *Physical Review D* **62**, 066004. *Preprint* hep-th/9908056.

Bogolubov, N. N. and Shirkov, D. V. (1980). *Introduction to the Theory of Quantized Fields*, 3rd ed. Wiley-Interscience, New-York.

Bolonek, K. and Kosinski, P. (2002). *Physics Letters B* **547**, 51.

Caetano, A. S. and Felder, G. (1999). A Path integral approach to the Kontsevich quantization formula math Q A/9902090.

Carey, R. M. *et al.* (1999). *Physical Review Letters* **82**, 1632.

Chaichian, M., Demichev, A., and Prešnajder, P. (2000a). *Nuclear Physics B* **567**, 360. *Preprint* hep-th/9812180.

Chaichian, M., Demichev, A., and Prešnajder, P. (2000b) *Journal of Mathematical Physics* **41**, 185. *Preprint* hep-th/9904132.

Chaichian, M., Demichev, A., and Prešnajder, P., Sheikh-Jabbari, M. M., and Tureanu, A. (2001a). *Nuclear Physics B* **611**, 383–402.

Chaichian, M., Sheikh-Jabbari, M. M., and Tureanu, A. (2001b). Hydrogen atom spectrum and the Lamb shift in noncommutative QED, *Physical Review Letters* **86**, 2716.

Chaichian, M., Sheikh-Jabbari, M. M., and Tureanu, A. (2000c). Space-time noncommutativity, discreteness of time and unitarity. *Preprint* hep-th/0007156.

Connes, A. (1994). *Noncommutative Geometry*, Academic Press, New York.

Connes, A., Douglas, M. R., and Schwarz, A. (1998). Noncommutative geometry and Matrix Theory: Compactification on tori. *Journal of High Energy Physics* **9802**, 003. *Preprint* hep-th/9711162.

Cutkosky, R. E. (1960). *Journal of Mathematical Physics* **1**, 429.

Czarnecki, A. and Marciano, W. J. (2001). *Physical Review D: Particles and Fields* **64**, 013014.

de Wit, B., Hoppe, J., and Nicolai, H. (1988). On the quantum mechanics of supermembranes. *Nuclear Physics B* **305**, 545.

Doplicher, S., Fredenhagen J., and Roberts, J. E. (1995). *Communications of Mathematical Physics* **172**, 187.

Douglas, M. R. and Nekrasov, N. A. (2001). *Review of Modern Physics* **73**, 977.

Dunne, G. V., Jackiw, R., and Trugenberger, C. A. (1990). "Topological" (Chern–Simons) quantum mechanics, *Physical Review D: Particles and Fields* **41**, 661.

Dunne, G. V. and Jackiw, R. (1993), "Peierls" substitution and Chern–Simons quantum mechanics, *Nuclear Physics C* **33**(Proc. Suppl.), 114. *Preprint* hep-th/9204057.

Duval, C. and Horvathy, P. A. (2000). The "Peierls" substitution and the exotic Galilei group. *Physics Letters B* **479**, 284. *Preprint* hep-th/0002233.

Efimov, G. V. (1977). *Nonlocal Interactions of Quantized Fields*, Nauka, Moscow.

Filk, T. (1996). *Physics Letters B* **376**, 53.

Floratos, E. G. and Nicolis, S. (2000). Quantum mechanics on the hypercube. *Preprint* hep-th/0006006.

Gamboa, J., Loewe, M., and Rojas, J. C. (2000). Noncommutative quantum mechanics. *Preprint* hep-th/0010220.

Gomis, J. and Mehen, T. (2000). *Nuclear Physics B* **591**, 265. *Preprint* hep-th/0005129.

Gracia-Bondi, J. M., Varilly, J. C., and Figueroa, V. (2000). *Elements of Noncommutative Geometry*, Birkhäuser, Boston.

Grosse, H., Klimčik, C., and Prešnajder, P. (1996a). *International Journal of Theoretical Physics* **35**, 231.

Grosse, H., Klimčik, C., and Prešnajder, P. (1996b). *Communications of Mathematical Physics* **178**, 507.

Grosse, H., Klimčik, C., and Prešnajder, P. (1997). *Communication of Mathematical Physics* **185**, 155.

Huang, W. H. (2001). Casimir effect on the radius stabilization of the noncommutative torus. *Physics Letters B* **497**, 317–322.

Hughes, V. W., and Kinoshita, T. (1999). *Review of Modern Physics* **71**, 5133.

Ishibashi, N., Kawai, H., Kitazawa, Y., and Tsuchiya, A. (1997). A large-N reduced model as superstring. *Nuclear Physics B* **498**, 467. *Preprint* hepth/9612115.

Jonke, L. and Meljanac, S. (2002). *Preprint* hep-th/0210042.

Kimura, Y. (2001). Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model. *Progress of Theoretical Physics* **106**, 445–469.

Kinoshita, T. (2001). *Preprint* hep-th/0101197.

Kontsevich, M. (1997). Deformation quantization of Poisson manifolds. *Preprint* q-alg/9709040.

Lehmann, H., Symanzik, K., and Zimmermann, W. (1955). *Zur Formulierung quantizierter Feldtheorien. Nuovo Cimento* **1**, 205.

Lehmann, H., Symanzik, K., and Zimmermann, W. (1957). The formulation of quantized field theories, II. *Nuovo Cimento* **6**, 319.

Lukierski, J., Stichel, P. C., and Zakrzewski, W. J. (1997). Galileaninvariant $(2+1)$ dimensional models with a Chern–Simons-like term and $D = 2$ noncommutative geometry. *Annalen de Physics* **260**, 224. *Preprint* hepth/9612017.

Madore, J. (1999). *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, Cambridge University Press, Cambridge, UK.

Morariu, B. and Polychronakos, A. P. (2001). Quantum mechanics on the noncommutative torus. *Nuclear Physics B* **610**, [P M], 531–544.

Nair, V. P. (2000). Quantum mechanics on a noncommutative brane in Matrix theory. *Preprint* hep-th/0008027.

Nair, V. P. and Polychronakos, A. P. (2000). Quantum mechanics on the noncommutative plane and sphere. *Preprint* hep-th/0011172.

Namsrai, Kh. (1986). *Nonlocal Quantum Field Theory and Stochastic Quantum Mechanics*, D. Reidel, Dordrecht, Holland.

Particle Data Group (2002). Review of particle physics, *Physical Review D* **66**(1), 1–958.

Polchinski, J. (1998). *String Theory, Vols. 1 and 2*, Cambridge University Press, Cambridge, UK.

Schomerus, V. (1999). *Journal of High Energy Physics*, **9906**, 030. *Preprint* hep-th/9903205.

Schwartz, L. (1957, 1959). *Theorie des Distributions, Vols. I and II*, Hermann, Paris.

Schwinger, J. (1948). *Physical Review* **73**, 416.

Seiberg, N., Susskind, L., and Toumbas, N. (2000). *Journal of High Energy Physics* **0006**, 044. *Preprint* hep-th/0005015.

Seiberg, N. and Witten, E. (1999). *Journal of High-Energy Physics* **9909**, 032. *Preprint* hep-th/9908142.

Snyder, H. (1947). *Physical Review* **71**, 38.

Susskind, L. (2001). The quantum Hall fluid and noncommutative Chern–Simons theory. *Preprint* hep-th/0101029.

Szabo, R. J. (in press). *Physics Reports*. *Preprint* hep-th/0109162.

‘t Hooft, G. and Veltman, M. (1972). Regularization and renormalization of Gauge Fields. *Nuclear Physics B* **44**, 189–213.

’t Hooft, G. and Veltman, M. (1973). Diagrammar, CERN, *Preprint*, CERN 73–9, Geneva.

Weinberg, J. S. (1995). *The Quantum Theory of Fields, Vol. 1: Foundations*, Cambridge University Press, Cambridge, UK.

Weinberg, J. S. (1972). *Gravitation and Cosmology; Principles and Applications of the General Theory of Relativity*, Wiley, New York.

Witten, E. (1996). Bound states of strings and p-branes, *Nuclear Physics B* **460**, 335. *Preprint*, hep-th/9510135.